Baroclinic Enhancements to the ADCIRC Finite Element Model
Version 35.03

J.J. Westerink
Department of Civil Engineering and Geological Sciences
University of Notre Dame
Notre Dame, IN 46556

Rick Luettich
University of North Carolina at Chapel Hill
Institute of Marine Sciences
3431 Arendell St.
Morehead City, NC 28557

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Rick Luettich
University of North Carolina at Chapel Hill
Institute of Marine Sciences
3431 Arendell St.
Morehead City, NC 28557

Joannes Westerink
Department of Civil Engineering and Geological Sciences
University of Notre Dame
Notre Dame, IN 46556

Both the vertically-integrated (ADCIRC-2DDI) and the fully three-dimensional (ADCIRC-3D) versions of ADCIRC solve a vertically-integrated continuity equation for water surface elevation. To avoid the spurious oscillations that are associated with a primitive Galerkin finite element formulation of this equation, ADCIRC utilizes the Generalized Wave Continuity Equation (GWCE) formulation. The weighted residual statement for the GWCE used in ADCIRC is:

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_i \right\rangle_\Omega + \left\langle \tau_o \frac{\partial \zeta}{\partial t}, \phi_i \right\rangle_\Omega + \left\langle gh \frac{\partial \zeta}{\partial x}, \frac{\partial \phi_i}{\partial x} \right\rangle_\Omega + \left\langle gh \frac{\partial \zeta}{\partial y}, \frac{\partial \phi_i}{\partial y} \right\rangle_\Omega =$$
$$\left\langle W_x, \frac{\partial \phi_i}{\partial x} \right\rangle_\Omega + \left\langle W_y, \frac{\partial \phi_i}{\partial y} \right\rangle_\Omega + \int_\Gamma \left[\frac{\partial Q_n}{\partial t} + \tau_o Q_n \right] \phi_i \, d\Gamma \tag{1}$$

where,

$$U,V \equiv \frac{1}{H} \int_{-h}^{\zeta} u, v\, dz$$

$$H \equiv \zeta + h$$

$$W_x \equiv \tau_o UH + U \frac{\partial \zeta}{\partial t} - UH \frac{\partial U}{\partial x} - VH \frac{\partial U}{\partial y} + fVH - \frac{g}{2} \frac{\partial \zeta^2}{\partial x} - H \frac{\partial \left[P_s / \rho_o - \alpha g \eta \right]}{\partial x}$$
$$+ \frac{\tau_{sx}}{\rho_o} - \frac{\tau_{bx}}{\rho_o} + M_x - D_x - B_x$$

$$W_y \equiv \tau_o VH + V\frac{\partial \zeta}{\partial t} - UH\frac{\partial V}{\partial x} - VH\frac{\partial V}{\partial y} - fUH - \frac{g}{2}\frac{\partial \zeta^2}{\partial y} - H\frac{\partial\left[P_s/\rho_o - \alpha g \eta\right]}{\partial y}$$

$$+\frac{\tau_{sy}}{\rho_o} - \frac{\tau_{by}}{\rho_o} + M_y - D_y - B_y$$

$$M_x \equiv E_h\left[\frac{\partial^2 UH}{\partial x^2} + \frac{\partial^2 UH}{\partial y^2}\right]$$

$$M_y \equiv E_h\left[\frac{\partial^2 VH}{\partial x^2} + \frac{\partial^2 VH}{\partial y^2}\right]$$

$$D_x \equiv \frac{\partial D_{uu}}{\partial x} + \frac{\partial D_{uv}}{\partial y}$$

$$D_y \equiv \frac{\partial D_{uv}}{\partial x} + \frac{\partial D_{vv}}{\partial y}$$

$$D_{uu} \equiv \int_{-h}^{\zeta}(u-U)(u-U)\,dz$$

$$D_{uv} \equiv \int_{-h}^{\zeta}(u-U)(v-V)\,dz$$

$$D_{vv} \equiv \int_{-h}^{\zeta}(v-V)(v-V)\,dz$$

$$B_x \equiv \int_{-h}^{\zeta} b_x\,dz$$

$$B_y \equiv \int_{-h}^{\zeta} b_y\,dz$$

$$b_x \equiv g\frac{\partial \int_{-h}^{\zeta}(\rho-\rho_o)/\rho_o\,dz}{\partial x}$$

$$b_y \equiv g\frac{\partial \int_{-h}^{\zeta}(\rho-\rho_o)/\rho_o\,dz}{\partial y}$$

ζ = free surface departure from still water
u,v = horizontal velocities
h = bathymetric water depth
ϕ_i = linear basis function
x,y,z = horizontal and vertical coordinates with $z = 0$ at the free surface
t = time coordinate
ρ = time and spatially varying density of water due to salinity and temperature variations
ρ_o = reference density of water
Q_n = positive inward normal flux per unit width along the boundary
τ_o = primitive continuity equation weighting parameter
τ_{sx}, τ_{sy} = imposed surface stresses

τ_{bx}, τ_{by} = bottom stress components, suitably defined
P_s = atmospheric pressure at the sea surface
η = Newtonian equilibrium tide potential
M_x, M_y = vertically-integrated horizontal stresses
E_h = horizontal eddy viscosity
D_x, D_y = momentum dispersion terms. ADCIRC-2DDI assumes $D_x, D_y = 0$.
B_x, B_y = vertically-integrated buoyancy terms
$\left\langle \Upsilon, \phi_i \right\rangle_\Omega \equiv \sum_{n=1}^{NE_i} \int_{\Omega_n} \Upsilon \phi_i \, d\Omega$ = horizontal integration over the elements surrounding node i
$\int_\Gamma [\ ] d\Gamma$ = boundary integral

ADCIRC uses the shallow water form of the momentum equations (including the Boussinesq and hydrostatic pressure approximations). ADCIRC-2DDI solves the vertically-integrated version of these equations while ADCIRC-3D solves the full three-dimensional set.

The weighted residual statements for the momentum equations used in ADCIRC-2DDI are:

$$\begin{aligned}&\left\langle \frac{\partial U}{\partial t}, \phi_i \right\rangle_\Omega + \left\langle \left(U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right), \phi_i \right\rangle_\Omega - \left\langle fV, \phi_i \right\rangle_\Omega = \\ &-\left\langle \frac{\partial [g\zeta + P_s - \alpha g \eta]}{\partial x}, \phi_i \right\rangle_\Omega + \left\langle \frac{\tau_{sx}}{H\rho_o}, \phi_i \right\rangle_\Omega - \left\langle \frac{\tau_{bx}}{H\rho_o}, \phi_i \right\rangle_\Omega + \left\langle \frac{M_x}{H}, \phi_i \right\rangle_\Omega - \left\langle \frac{B_x}{H}, \phi_i \right\rangle_\Omega\end{aligned} \tag{2}$$

$$\begin{aligned}&\left\langle \frac{\partial V}{\partial t}, \phi_i \right\rangle_\Omega + \left\langle \left(U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} \right), \phi_i \right\rangle_\Omega + \left\langle fU, \phi_i \right\rangle_\Omega = \\ &-\left\langle \frac{\partial [g\zeta + P_s - \alpha g \eta]}{\partial y}, \phi_i \right\rangle_\Omega + \left\langle \frac{\tau_{sy}}{H\rho_o}, \phi_i \right\rangle_\Omega - \left\langle \frac{\tau_{by}}{H\rho_o}, \phi_i \right\rangle_\Omega + \left\langle \frac{M_y}{H}, \phi_i \right\rangle_\Omega - \left\langle \frac{B_y}{H}, \phi_i \right\rangle_\Omega\end{aligned} \tag{3}$$

where,

$\tau_{bx} \equiv C_d \left(U^2 + V^2 \right)^{1/2} U$ and $\tau_{by} \equiv C_d \left(U^2 + V^2 \right)^{1/2} V$ = quadratic friction law

The momentum equations use a different rule for horizontal integration than the GWCE:

$\left\langle \Upsilon, \phi_i \right\rangle_\Omega \equiv \frac{1}{NE_i} \sum_{n=1}^{NE_i} \frac{3}{A_n} \int_{\Omega_n} \Upsilon \phi_i \, d\Omega$ for terms containing horizontal gradients

$\left\langle \Upsilon, \phi_i \right\rangle_\Omega \equiv \frac{1}{NE_i} \sum_{n=1}^{NE_i} \frac{3}{A_n} \Upsilon_i \int_{\Omega_n} \phi_i \, d\Omega = \Upsilon_i$ for terms that do not contain horizontal gradients.

The weighted residual statements for the momentum equations used in ADCIRC-3D are:

$$\left\langle\left\langle\frac{\partial u}{\partial t},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z+\left\langle\left\langle\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial \sigma}\right),\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z-\left\langle\left\langle fv,\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z=$$
$$-\left\langle\left\langle\frac{\partial\left[g\zeta+P_s/\rho_o-\alpha g\eta\right]}{\partial x},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z+\left\langle\left\langle\left(\frac{a-b}{H}\right)^2E_z\frac{\partial u}{\partial\sigma},\phi_i\right\rangle_\Omega,\frac{\partial\psi_k}{\partial\sigma}\right\rangle_Z \tag{4}$$
$$+\left\langle\left\langle\frac{\tau_{sx}}{\rho_o},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z-\left\langle\left\langle\frac{\tau_{bx}}{\rho_o},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z+\left\langle\left\langle m_x,\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z-\left\langle\left\langle b_x,\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z$$

$$\left\langle\left\langle\frac{\partial v}{\partial t},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z+\left\langle\left\langle\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial \sigma}\right),\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z+\left\langle\left\langle fu,\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z=$$
$$-\left\langle\left\langle\frac{\partial\left[g\zeta+P_s/\rho_o-\alpha g\eta\right]}{\partial y},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z+\left\langle\left\langle\left(\frac{a-b}{H}\right)^2E_z\frac{\partial v}{\partial\sigma},\phi_i\right\rangle_\Omega,\frac{\partial\psi_k}{\partial\sigma}\right\rangle_Z \tag{5}$$
$$+\left\langle\left\langle\frac{\tau_{sy}}{\rho_o},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z-\left\langle\left\langle\frac{\tau_{by}}{\rho_o},\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z+\left\langle\left\langle m_y,\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z-\left\langle\left\langle b_y,\phi_i\right\rangle_\Omega,\psi_k\right\rangle_Z$$

where,

E_z = vertical eddy viscosity that can be specified or computed using a MY 2.5 closure

$\left\langle\Upsilon,\psi_k\right\rangle_Z\equiv\sum_{n=1}^{2}\int_{Z_n}\Upsilon\psi_k\,dz$ = vertical integration over the elements on either side of vertical node k

$\left\langle\Upsilon,\phi_i\right\rangle_\Omega$ = the horizontal integration as described for the 2DDI momentum equations.

Prior to ADCIRC version 35, the buoyancy terms shown above have not been included in the GWCE or either the 2DDI or 3D momentum equations. These have now been implemented for both the 2DDI and 3D versions of the code to provide ADCIRC with the capability for baroclinic forcing. We note that to avoid problems with the baroclinic pressure gradient near areas of steep topography, the density field is interpolated in the vertical and the buoyancy terms (i.e., b_x, b_y in Eqs (4)-(5)) are evaluated along level coordinate surfaces.

An initial test case was run to exercise the buoyancy terms in both 2D and 3D. A closed rectangular basin was set up with dimensions 48km long x 16km wide x 10m deep. Nodes were evenly spaced in the horizontal with $\Delta x=\Delta y=1$km. For the 3D tests, twenty-one sigma levels were used over the vertical. The flow was specified as initially at rest with a vertically uniform density field that increased linearly from 1000 kg/m^3 at one end to 1036 kg/m^3 at the other. This density field was held constant for the duration of the simulation. Upon initiation of the run, the horizontal density gradient created a flow toward the low-density end of the channel.

At steady state in two dimensions a balance will exist between the density gradient and the free surface slope with the depth-averaged velocity equal to zero. An analytical solution can easily be found for the steady state condition:

$$\frac{\partial \zeta}{\partial x} = -\frac{h}{2}\frac{\partial \rho}{\partial x}$$

For $\partial \rho / \partial x = 7.5x10^{-7}$ the free surface slopes from 0.09m at the low-density end of the channel to –0.09m at the high-density end of the channel. Figure 1 presents a time series of elevation at the low-density end of the channel and indicates oscillatory, asymptotic convergence to the analytical solution. Oscillations occur due to the relatively week influence of bottom friction in the problem (C_d=0.025). Convergence can be accelerated by using a larger bottom friction coefficient.

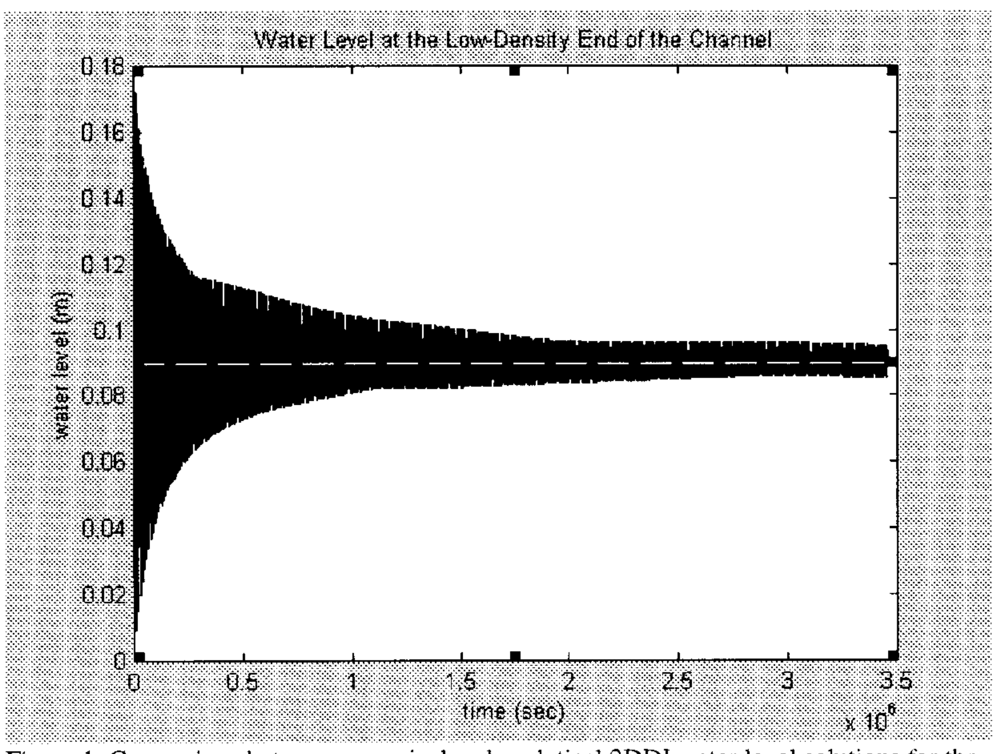

Figure 1. Comparison between numerical and analytical 2DDI water level solutions for the density driven test case. The transient ADCIRC solution is shown by the solid line and the steady state analytical solution is shown by the dashed line.

At steady state in three dimensions a balance will exist between the density gradient, the free surface slope and friction. The baroclinic pressure gradient is greatest at the bottom of the channel and consequently a flow is driven in the lower part of the water column toward the low-density end of the channel. The analytical solution for a linearized bottom slip and vertically constant E_z is

$$u = -\frac{g}{6E_z}\frac{\partial\rho}{\partial x}\left(z^3 + \frac{h^3}{4}\right) + \frac{g}{2E_z}\frac{\partial\zeta}{\partial x}\left(z^2 - \frac{h^2}{3}\right)$$

$$\frac{\partial\zeta}{\partial x} = -\frac{h}{2}\frac{\partial\rho}{\partial x}\frac{\left(1+\dfrac{kh}{4E_z}\right)}{\left(1+\dfrac{kh}{3E_z}\right)}$$

Figure 2 presents the vertical profile of along channel velocity in the middle of the channel at approximately steady state conditions ($U = 2x10^{-5} m/s$) for E_z=0.05 m^2/s, a linear bottom slip coefficient k=0.005 s^{-1} and a longitudinal density gradient $\partial\rho/\partial x = 7.5x10^{-7}$. As indicated by Figure 2, the numerical velocity profile is nearly identical to the corresponding analytical solution.

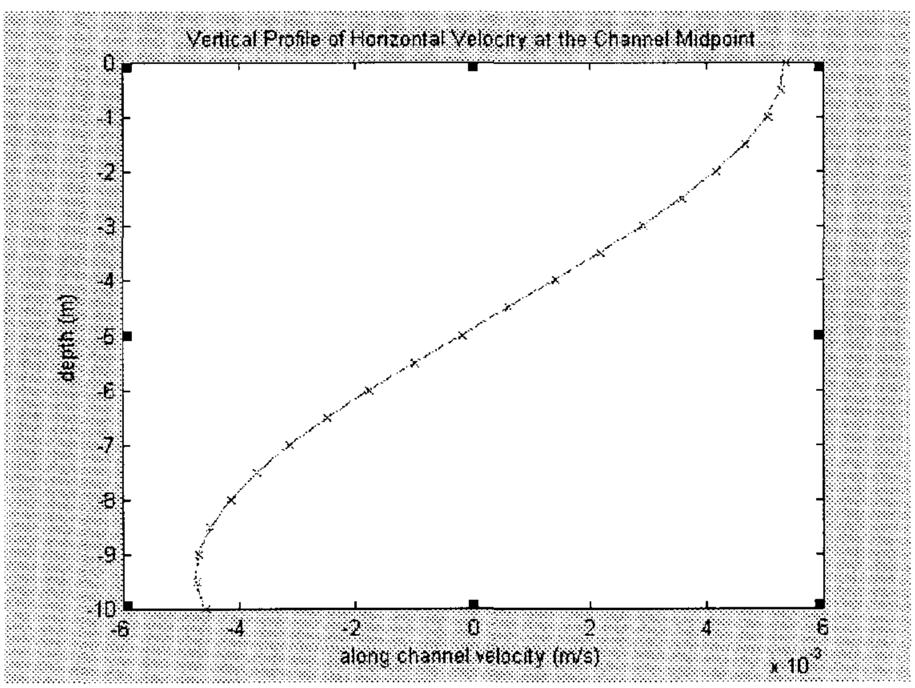

Figure 2. Comparison between steady state numerical and analytical 3D solutions for the density driven test case. The solid line indicates the analytical solution and the "x" symbols indicate the ADCIRC solution.