DISCONTINUOUS AND COUPLED CONTINUOUS/DISCONTINUOUS GALERKIN METHODS FOR THE SHALLOW WATER EQUATIONS

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Abstract. We consider the approximation of a simplified model of the depth-averaged two dimensional shallow water equations by two approaches. In both approaches, a discontinuous Galerkin (DG) method is used to approximate the continuity equation. In the first approach, a continuous Galerkin method is used for the momentum equations. In the second approach a particular DG method, the nonsymmetric interior penalty Galerkin method (NIPG), is used to approximate momentum. *A priori* error estimates are derived and numerical results are presented for both approaches.

Key words. shallow water equations, Galerkin finite element method, discontinuous Galerkin method

1. Introduction. Simulation of flow in shallow water systems can serve numerous purposes. Examples include modeling environmental effects of dredging and commercial activities on fisheries and coastal wildlife, remediation of contaminated bays and estuaries for the purposes of improving water quality, modeling the effects of storm surges due to tropical storms and hurricanes, and studying freshwater-saltwater interactions.

The shallow water equations model flow in domains whose characteristic wave length in the horizontal is much larger than the water depth. Integrating the three-dimensional mass and momentum equations over the depth, and applying the kinetic boundary conditions at the free surface, one obtains the two dimensional depth-integrated shallow water equations (SWE); see [35]. The SWE consist of a first order hyperbolic continuity equation for the water elevation, coupled to momentum equations for the horizontal depth-averaged velocities. This system is referred to as the primitive form of the shallow water equations. These equations are often solved on domains with fairly irregular (land) boundaries. Furthermore, to avoid spurious boundary effects, it is often desirable to extend the domain away from the shore into deeper waters [38, 5].

Motivated in part by the desire to model flow in complex domains, various finite element approaches have been developed for the SWE over the past two decades; see, for example, [31, 27, 26, 34, 41]. Much of this effort has been directed at deriving a finite element method which is stable and nonoscillatory under highly varying flow regimes, including advection dominant flows. As noted in [31], a straightforward use of equal order approximating spaces for elevation and velocity in the primitive SWE can lead to spurious spatial oscillations. Approaches based on mixed interpolation spaces [27] have met with limited success. A more widespread approach has been to replace the first order hyperbolic elevation equation with a second order hyperbolic "wave continuity equation," first proposed in [31]. This approach has served as the basis for numerous finite element studies, see for example, [28, 25, 24, 22, 23, 30, 36, 37, 38, 6, 5, 29, 39], and was analyzed in [9, 10].

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The finite element methods mentioned above are based on continuous approximating spaces. The wave continuity formulation sacrifices the primitive continuity equation, thus the primitive form is no longer satisfied in a discrete sense. In recent years, finite element methods based on discretizing the primitive form of the SWE using discontinuous approximating spaces have been studied [2, 11, 1]. This discontinuous Galerkin (DG) approach has several appealing features; in particular, the ability to incorporate upwinding and stability post-processing into the solution to model highly advective flows, the ability to use different polynomial orders of approximation in different parts of the domain (and for different variables, if so desired), and the ability to easily use nonconforming meshes (e.g., with hanging nodes). Moreover, the DG method is "locally conservative," that is, the continuity equation relating the change in water elevation to water flux, is satisfied in a weak sense element by element. This latter property is useful when coupling the SWE to a transport equation for modeling, for example, contaminant migration [20]. DG methods have proven adept at modeling hyperbolic equations [17, 16, 15, 13, 19, 4], advection-diffusion [3, 18, 12, 21] and pure diffusion equations [40, 8, 32, 33]. See also [14] for a more thorough discussion on the history of DG methods.

In this paper, we will develop and analyze two approaches for shallow water flow modeling based on discontinuous and continuous approximating spaces. In the first approach, we discretize the primitive continuity equation using a DG method, coupled to a continuous finite element approximation of the momentum equations. This approach is useful when local conservation is important, and uses discontinuous approximations for the hyperbolic continuity equation, while allowing for the momentum equation to be approximated using more traditional continuous functions. In the second approach, we discretize both equations using DG methods. For the momentum equation, we use a particular DG method called the nonsymmetric interior penalty Galerkin (NIPG) method, developed in [33]. This approach allows for the flexibility of the DG method to be applied to both continuity and momentum, if so desired.

The paper is organized as follows. In the following section, we describe the mathematical model and define notation. Section three contains the description of the scheme whereby we discretize the continuity equation by a DG method and the momentum equation by a continuous Galerkin finite element method. *A priori* error estimates are then proven for this approach. In section four, the continuity equation is again discretized using DG while the momentum equation is discretized by the NIPG formulation. Error estimates for this formulation are also derived. Section four contains some numerical results comparing the two methods and the final section contains concluding remarks.

2. Problem definition. The depth-averaged shallow water equations are derived from the three dimensional incompressible Navier-Stokes equations under the assumptions of a long horizontal wavelength and a hydrostatic pressure distribution. The system consists of the primitive continuity equation and momentum equations. Unknown variables are depth-averaged elevation $\zeta = \zeta(\mathbf{x}, t)$ and velocity $\boldsymbol{u} = \boldsymbol{u}(\mathbf{x}, t)$.

Let Ω be a bounded domain in $\mathbb{R}^2$, with Lipschitz boundary $\partial\Omega$, where $\boldsymbol{n}$ is the fixed unit outward normal to $\partial\Omega$. For the continuity equation, we decompose the boundary of the domain into an inflow portion $\partial\Omega_{\mathrm{I}}$ and an outflow portion $\partial\Omega_{\mathrm{O}}$ such that $\partial\Omega = \partial\Omega_{\mathrm{I}} \cup \partial\Omega_{\mathrm{O}}$, where $\partial\Omega_{\mathrm{I}} = \{x \in \partial\Omega : \boldsymbol{u} \cdot \mathbf{n} < 0\}$ and $\partial\Omega_{\mathrm{O}} = \{x \in \partial\Omega : \boldsymbol{u} \cdot \mathbf{n} \geq 0\}$. For the momentum equation, we assume Dirichlet boundary conditions on $\boldsymbol{u}$ are specified everywhere on the boundary of the domain.

Consider the following simplified form of the shallow water equations: find ζ and

Symbol	Description
ζ	Elevation of air-water interface from mean sea level
$\boldsymbol{u}$	Vertically averaged horizontal velocity field
g	Acceleration due to gravity
ν	Vertically averaged turbulent viscosity
$\boldsymbol{f}$	Body forces

TABLE 1
SWE variables and parameters

$\boldsymbol{u}$ such that

$$\partial_t \zeta + \nabla \cdot (\boldsymbol{u}\zeta) = 0 \qquad (\mathbf{x}, t) \in \Omega,\ t > 0, \tag{1}$$

$$\partial_t \boldsymbol{u} + g\nabla\zeta - \nu\Delta\boldsymbol{u} = \boldsymbol{f} \qquad (\mathbf{x}, t) \in \Omega,\ t > 0. \tag{2}$$

This simplified model contains the primary coupling between the two equations. We will consider boundary and initial conditions of the form

$$\zeta = \hat{\zeta} \qquad \text{on } \partial\Omega_{\mathrm{I}}, \tag{3}$$

$$\zeta(\mathbf{x}, 0) = \zeta_0 \qquad \text{on } \Omega, \tag{4}$$

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \qquad \text{on } \partial\Omega, \tag{5}$$

$$\boldsymbol{u}(\mathbf{x}, 0) = \boldsymbol{u}_0 \qquad \text{on } \Omega. \tag{6}$$

Here ν is assumed to be a positive constant. Table 2.1 contains the variable and parameter definitions for reference.

2.1. Notation and Function Space Properties. Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of finite element partitions of Ω such that no element Ω_e crosses $\partial\Omega$. We assume each element Ω_e has a element diameter h_e, with h being the maximal element diameter. We also assume each element Ω_e is affinely equivalent to one of several reference elements [7]. Further assumptions on $\mathcal{T}_h$ will be given below. Let $\mathcal{P}^k(\Omega_e)$ denote the space of complete polynomials of degree $k \geq 1$, defined on Ω_e.

For any function $v \in H^1(\Omega_e)$, for each element Ω_e, we denote its trace on interior edges γ_i by $v^{\pm}$:

$$v^-(\mathbf{x}) = \lim_{s \to 0-} v(\mathbf{x} + s\boldsymbol{n}_i), \qquad v^+(\mathbf{x}) = \lim_{s \to 0+} v(\mathbf{x} + s\boldsymbol{n}_i),$$

then define

$$\overline{v} = \frac{1}{2}(v^+ + v^-), \qquad [v] = v^- - v^+,$$

where $\mathbf{x} \in \gamma_i$ and $\boldsymbol{n}_i$ denotes a fixed unit vector normal to γ_i. Let $\sum_i$ denote summation over all interior element edges γ_i.

We will use the $L^2(R)$ inner product notation $(\cdot,\cdot)_R$ for domains $R \in \mathbb{R}^2$, and the notation $\langle u, v\rangle_R$ to denote integration over one-dimensional surfaces. Let $||\cdot||_R$ denote the $L^2(R)$ norm on any spatial region R. It will be understood that if $g \in L^2(\Omega_e)$, then

$$||g||_\Omega^2 \equiv \sum_e ||g||_{\Omega_e}^2.$$

Norms in other Sobolev spaces $W(R)$ are denoted by $\|\cdot\|_{W(R)}$. Furthermore, for $g = g(x,t)$,

$$||g||_{L^\infty(0,T;L^2(R))} = \max_{0\le t\le T} ||g(\cdot,t)||_{L^2(R)},$$
$$||g||^2_{L^2(0,T;L^2(R))} = \int_0^T ||g(\cdot,t)||^2_{L^2(R)} dt.$$

In our analysis, we will use the following well-known trace theorem [7].

THEOREM 2.1. *Suppose that Ω_e has a Lipschitz boundary. Then, there is a constant K_e^t such that*

$$\|v\|_{L^2(\partial\Omega_e)} \le K_e^t \|v\|^{1/2}_{L^2(\Omega_e)} \|v\|^{1/2}_{H^1(\Omega_e)} \qquad \forall v \in H^1(\Omega_e).$$

Define

$$K^t = \max_e K_e^t.$$

We will also make use of Young's inequality: for real numbers a, b and $\epsilon > 0$,

$$ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2.$$

3. The discontinuous and continuous Galerkin formulation. In this section, we will discretize the primitive continuity equation by a discontinuous Galerkin method and the corresponding momentum equations by a standard Galerkin finite element method. We will assume that the partitions $\mathcal{T}_h$ are quasi-uniform and conforming; i.e., element edges align with neighboring element edges.

Multiply equation (1) by arbitrary, smooth test functions $v \in H^1(\Omega_e)$ and integrate by parts over each element Ω_e to obtain

$$(\partial_t\zeta, v)_{\Omega_e} - (\boldsymbol{u}\,\zeta, \nabla v)_{\Omega_e} + \langle \zeta\, \boldsymbol{u}\cdot\boldsymbol{n}_e, v\rangle_{\partial\Omega_e} = 0, \qquad (7)$$

where $\boldsymbol{n}_e$ denotes a fixed unit normal to each edge $\partial\Omega_e$.

On each Ω_e, we approximate ζ in a space $\mathcal{S}^k(\Omega_e)$, where $\mathcal{P}^k(\Omega_e) \subset \mathcal{S}^k(\Omega_e)$, and such that if we define

$$V_h = \{v : \Omega \to I\!R : v|_{\Omega_e} \in \mathcal{S}^k(\Omega_e)\}, \qquad (8)$$

then V_h^c, the space of continuous, piecewise polynomials of degree k, is contained in V_h, that is,

$$V_h^c = V_h \cap \mathcal{C}^0(\Omega) \ne \emptyset.$$

Multiply equation (2) by $\boldsymbol{w} \in (H_0^1(\Omega))^2$ and integrate by parts over the domain to obtain

$$(\partial_t\boldsymbol{u}, \boldsymbol{w})_\Omega + (g\nabla\zeta, \boldsymbol{w})_\Omega - \sum_i \langle g\,[\zeta], \boldsymbol{w}\cdot\boldsymbol{n}_i\rangle_{\gamma_i} + (\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega. \qquad (9)$$

Note that the stabilization term $\sum_i \langle g[\zeta], \boldsymbol{w}\cdot\boldsymbol{n}_i\rangle_{\gamma_i}$ is actually zero since we are assuming our true solution sufficiently smooth to be continuous.

We approximate $\boldsymbol{u}$ in the finite-dimensional subspace $\boldsymbol{W}_h \subset (H^1(\Omega))^2 \cap \{\boldsymbol{u} : \boldsymbol{u} = \hat{\boldsymbol{u}} \text{ on } \partial\Omega\}$, consisting of continuous, piecewise polynomials of degree k. That is, each component of $\boldsymbol{u}$ is in V_h^c. Let $\boldsymbol{W}_{0,h}$ be the corresponding subspace of $(H_0^1(\Omega))^2$.

Approximate $\zeta(\cdot,t)$ by $Z(\cdot,t) \in V_h$, and $\boldsymbol{u}(\cdot,t)$ by $\boldsymbol{U}(\cdot,t) \in \boldsymbol{W}_h$. Sum (7) over all elements Ω_e, and let the value of ζ across inner element boundaries be approximated by the upwind value $Z^\uparrow$:

$$\zeta \approx Z^\uparrow = \begin{cases} Z^-, & \boldsymbol{U}\cdot\boldsymbol{n}_i > 0 \\ Z^+, & \boldsymbol{U}\cdot\boldsymbol{n}_i \le 0 \end{cases} \qquad \text{on } \gamma_i.$$

At $t = 0$ define $Z(\cdot,0) = Z_0 \in V_h$ and $\boldsymbol{U}(\cdot,0) = \boldsymbol{U}_0 \in \boldsymbol{W}_h$ by

$$(Z_0 - \zeta_0, v)_\Omega = 0, \qquad \forall\, v \in V_h, \tag{10}$$

$$(\boldsymbol{U}_0 - \boldsymbol{u}_0, \boldsymbol{w})_\Omega = 0, \qquad \forall\, w \in \boldsymbol{W}_{0,h}. \tag{11}$$

The discrete weak formulation is: for each $t > 0$, find $(Z, \boldsymbol{U}) \in V_h \times \boldsymbol{W}_h$ satisfying $\forall\, v \in V_h$ and $\forall\, \boldsymbol{w} \in \boldsymbol{W}_{0,h}$:

$$\sum_e (\partial_t Z, v)_{\Omega_e} - \sum_e (\boldsymbol{U}\, Z, \nabla v)_{\Omega_e} + \sum_i \langle Z^\uparrow \boldsymbol{U}\cdot\boldsymbol{n}_i, [v]\rangle_{\gamma_i} + \langle Z\, \hat{\boldsymbol{u}}\cdot\boldsymbol{n}, v\rangle_{\partial\Omega_{\mathrm{O}}} = -\langle \hat{\zeta}\, \hat{\boldsymbol{u}}\cdot\boldsymbol{n}, v\rangle_{\partial\Omega_{\mathrm{I}}}, \tag{12}$$

$$(\partial_t \boldsymbol{U}, \boldsymbol{w})_\Omega + \sum_e (g\nabla Z, \boldsymbol{w})_{\Omega_e} - \sum_i \langle g\,[Z], \boldsymbol{w}\cdot\boldsymbol{n}_i\rangle_{\gamma_i} + (\nu\nabla\boldsymbol{U}, \nabla\boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega. \tag{13}$$

3.1. An *a priori* error estimate. Define $\zeta_I(\cdot,t) \in V_h^c$ to be the continuous interpolant of ζ. Define the parabolic projection $\Pi\boldsymbol{u}(\cdot,t) \in \boldsymbol{W}_h$ such that

$$(\partial_t(\Pi\boldsymbol{u} - \boldsymbol{u}), \boldsymbol{w})_\Omega + (\nu\nabla(\Pi\boldsymbol{u} - \boldsymbol{u}), \nabla\boldsymbol{w})_\Omega = 0 \;\; \forall \boldsymbol{w} \in \boldsymbol{W}_{0,h}(\Omega),$$

with $\Pi u(\cdot,0)$ equal to the L^2 projection of $\boldsymbol{u}_0$ into $\boldsymbol{W}_h$; that is,

$$(\Pi u(\cdot,0) - \boldsymbol{u}_0, \boldsymbol{w})_\Omega = 0, \quad \boldsymbol{w} \in \boldsymbol{W}_{0,h}(\Omega).$$

In our analysis, we will use a standard inverse inequality, valid for continous (and discontinuous) piecewise polynomials on quasi-uniform triangulations [7].

THEOREM 3.1. *Let $v \in V_h$. Then,*

$$||v||_{H^1(\Omega)} \le K^i\, h^{-1} ||v||_{L^2(\Omega)}, \tag{14}$$

where K^i is independent of h.

Our estimate will rely on certain smoothness of the solutions. We will assume the following constants are finite:

$$K_\zeta \equiv \int_0^T \left[||\partial_t\zeta||^2_{H^k(\Omega)} + ||\zeta||^2_{H^{k+1}(\Omega)} \right] dt + ||\zeta_0||^2_{H^k(\Omega)}, \tag{15}$$

$$K_{\boldsymbol{u}} \equiv \int_0^T \left[||\partial_t\boldsymbol{u}||^2_{H^k(\Omega)} + ||\boldsymbol{u}||^2_{H^{k+1}(\Omega)} \right] dt + ||\boldsymbol{u}_0||^2_{H^k(\Omega)}, \tag{16}$$

$$K^*_{\boldsymbol{u}} \equiv ||\boldsymbol{u}||_{L^\infty(0,T;W_1^\infty(\Omega))}, \tag{17}$$

$$K^*_\zeta \equiv ||\zeta_I||_{L^\infty(0,T;W_1^\infty(\Omega))}. \tag{18}$$

Finally, we will assume that a constant $K_Z^* \geq 2K_\zeta^*$, independent of h, exists such that the finite element solution Z satisfies

$$||Z||_{L^\infty(0,T;L^\infty(\Omega))} \leq K_Z^*. \tag{19}$$

For $k > 1$, we will prove inductively that for h sufficiently small, our estimate does not in fact depend on K_Z^*.

Define

$$\begin{aligned} e_\zeta &= Z - \zeta_I, & \boldsymbol{e}_u &= \boldsymbol{U} - \Pi\boldsymbol{u}, \\ \theta_\zeta &= \zeta - \zeta_I, & \boldsymbol{\theta}_u &= \boldsymbol{u} - \Pi\boldsymbol{u}, \\ e_\zeta^\uparrow &= Z^\uparrow - \zeta_I. \end{aligned}$$

THEOREM 3.2. *For $\boldsymbol{u}$,ζ sufficiently smooth, the scheme (12)-(13) satisfies the error estimate*

$$|\!|\!| (e_\zeta, \boldsymbol{e}_u) |\!|\!| \leq K_1 h^k, \tag{20}$$

where

$$\begin{aligned} 2|\!|\!| (e_\zeta, \boldsymbol{e}_u) |\!|\!|^2 = \|e_\zeta(T)\|_\Omega^2 + \|\boldsymbol{e}_u(T)\|_\Omega^2 \\ + \int_0^T \sum_i \langle |\boldsymbol{U}\cdot\boldsymbol{n}_i|, [e_\zeta]^2\rangle_{\gamma_i} dt + \int_0^T \langle |\hat{\boldsymbol{u}}\cdot\boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega_O} dt \\ + \int_0^T \langle |\boldsymbol{U}\cdot\boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega_I} dt + \int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u\|_\Omega^2 dt, \end{aligned}$$

and K_1 is a constant independent of h and k, but dependent on g, ν, K^i, K^t, K_ζ, $K_{\boldsymbol{u}}$, K_ζ^*, $K_{\boldsymbol{u}}^*$ and K_Z^*. For $k > 1$ and h sufficiently small, the dependence of K_1 on K_Z^* can be removed.

Proof. Standard approximation results for ζ_I and $\Pi\boldsymbol{u}$ give

$$\int_0^T [||\partial_t\theta_\zeta||_\Omega^2 + ||\theta_\zeta||_{H^1(\Omega)}^2 + h^{-2}||\theta_\zeta||_\Omega^2]dt + ||\theta_\zeta(0)||_\Omega^2 \leq K(K_\zeta)h^{2k}, \tag{21}$$

and

$$\int_0^T [h^{-2}||\boldsymbol{\theta}_u||_\Omega^2 + ||\boldsymbol{\theta}_u||_{H^1(\Omega)}^2]dt \leq K(\nu, K_{\boldsymbol{u}})h^{2k}. \tag{22}$$

Subtract the weak formulation from the corresponding discrete formulation (12)-(13), and integrate in time from 0 to T. Incorporate $\Pi\boldsymbol{u}$ and ζ_I, and take $(v, \boldsymbol{w}) =$

$(e_\zeta, \boldsymbol{e}_u)$ to obtain

$$
\begin{aligned}
\int_0^T \sum_e (\partial_t e_\zeta, e_\zeta)_{\Omega_e} dt - \int_0^T \sum_e (\boldsymbol{U}\, e_\zeta, \nabla e_\zeta)_{\Omega_e} dt + \int_0^T \sum_i \langle e_\zeta^\uparrow \boldsymbol{U} \cdot \boldsymbol{n}_i, [e_\zeta] \rangle_{\gamma_i} dt \\
+ \int_0^T \langle e_\zeta\, \hat{\boldsymbol{u}} \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_O} dt + \int_0^T (\partial_t \boldsymbol{e}_u, \boldsymbol{e}_u)_\Omega dt + \int_0^T \sum_e (g \nabla e_\zeta, \boldsymbol{e}_u)_{\Omega_e} dt \\
- \int_0^T \sum_i \langle g\, [e_\zeta], \boldsymbol{e}_u \cdot \boldsymbol{n}_i \rangle_{\gamma_i} dt + \int_0^T (\nu \nabla \boldsymbol{e}_u, \nabla \boldsymbol{e}_u)_\Omega dt \\
= \int_0^T \sum_e (\partial_t \theta_\zeta, e_\zeta)_{\Omega_e} dt - \int_0^T \sum_e (\boldsymbol{u}\, \zeta - \boldsymbol{U}\, \zeta_I, \nabla e_\zeta)_{\Omega_e} dt \\
+ \int_0^T \sum_i \langle \zeta\, \boldsymbol{u} \cdot \boldsymbol{n}_i - \zeta_I\, \boldsymbol{U} \cdot \boldsymbol{n}_i, [e_\zeta] \rangle_{\gamma_i} dt \\
+ \int_0^T \langle \theta_\zeta\, \hat{\boldsymbol{u}} \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_O} dt + \int_0^T (g \nabla \theta_\zeta, \boldsymbol{e}_u)_\Omega dt. \quad (23)
\end{aligned}
$$

We first consider the left hand side of this equation. The first term and fifth terms can be written

$$
\begin{aligned}
\int_0^T \sum_e (\partial_t e_\zeta, e_\zeta)_{\Omega_e} dt &= \frac{1}{2} \|e_\zeta(T)\|_\Omega^2 - \frac{1}{2} \|e_\zeta(0)\|_\Omega^2, \\
&\geq \frac{1}{2} \|e_\zeta(T)\|_\Omega^2 - K(K_\zeta) h^{2k}, \quad (24) \\
\int_0^T \sum_e (\partial_t \boldsymbol{e}_u, \boldsymbol{e}_u)_{\Omega_e} dt &= \frac{1}{2} \|\boldsymbol{e}_u(T)\|_\Omega^2, \quad (25)
\end{aligned}
$$

since $\boldsymbol{e}_u(0) = 0$.

Integrate the second term by parts to obtain

$$
\begin{aligned}
- \int_0^T \sum_e (\boldsymbol{U}\, e_\zeta, \nabla e_\zeta)_{\Omega_e} dt = \frac{1}{2} \int_0^T \sum_e (\nabla \cdot \boldsymbol{U}, e_\zeta^2)_{\Omega_e} dt \\
- \frac{1}{2} \int_0^T \sum_i \langle \boldsymbol{U} \cdot \boldsymbol{n}_i, [e_\zeta^2] \rangle_{\gamma_i} dt - \frac{1}{2} \int_0^T \langle \boldsymbol{U} \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega} dt \quad (26)
\end{aligned}
$$

Note, however, that the jump term above can be combined with the third term in (23) yielding

$$
\int_0^T \sum_i \langle e_\zeta^\uparrow \boldsymbol{U} \cdot \boldsymbol{n}_i, [e_\zeta] \rangle_{\gamma_i} dt - \frac{1}{2} \int_0^T \sum_i \langle \boldsymbol{U} \cdot \boldsymbol{n}_i, [e_\zeta^2] \rangle_{\gamma_i} dt = \frac{1}{2} \int_0^T \sum_i \langle |\boldsymbol{U} \cdot \boldsymbol{n}_i|, [e_\zeta]^2 \rangle_{\gamma_i} dt. \quad (27)
$$

The boundary term can be combined with the fourth term in (23) to obtain

$$\begin{aligned}
&\int_0^T \langle \hat{\boldsymbol{u}} \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{O}}} dt - \frac{1}{2}\int_0^T \langle \boldsymbol{U} \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega} dt \\
&\quad = \int_0^T \langle \hat{\boldsymbol{u}} \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{O}}} dt - \frac{1}{2}\int_0^T \langle \boldsymbol{U} \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{I}}} dt - \frac{1}{2}\int_0^T \langle \Pi\boldsymbol{u} \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{O}}} dt \\
&\quad = \frac{1}{2}\int_0^T \langle |\hat{\boldsymbol{u}} \cdot \boldsymbol{n}|, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{O}}} dt. + \frac{1}{2}\int_0^T \langle |\boldsymbol{U} \cdot \boldsymbol{n}|, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{I}}} dt \\
&\qquad + \frac{1}{2}\int_0^T \langle (\hat{\boldsymbol{u}} - \Pi\boldsymbol{u}) \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{O}}} dt. \qquad (28)
\end{aligned}$$

Integrate the sixth term by parts and combine with the seventh term to obtain

$$\int_0^T \sum_e (g\nabla e_\zeta, \boldsymbol{e}_u)_{\Omega_e} dt - \int_0^T \sum_i \langle g\,[e_\zeta], \boldsymbol{e}_u \cdot \boldsymbol{n}_i \rangle_{\gamma_i} dt = -\int_0^T \sum_e (g\, e_\zeta, \nabla \cdot \boldsymbol{e}_u)_{\Omega_e} dt. \quad (29)$$

The eighth term can be written

$$\int_0^T (\nu \nabla \boldsymbol{e}_u, \nabla \boldsymbol{e}_u)_\Omega dt = \int_0^T \|\nu^{1/2} \nabla \boldsymbol{e}_u\|_\Omega^2 dt. \quad (30)$$

Combining each of these results into equation (23) yields

$$\begin{aligned}
&\frac{1}{2}\|e_\zeta(T)\|_\Omega^2 + \frac{1}{2}\|\boldsymbol{e}_u(T)\|_\Omega^2 \\
&\qquad + \frac{1}{2}\int_0^T \sum_i \langle |\boldsymbol{U} \cdot \boldsymbol{n}_i|, [e_\zeta]^2 \rangle_{\gamma_i} dt + \frac{1}{2}\int_0^T \langle |\hat{\boldsymbol{u}} \cdot \boldsymbol{n}|, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{O}}} dt \\
&\qquad + \frac{1}{2}\int_0^T \langle |\boldsymbol{U} \cdot \boldsymbol{n}|, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{I}}} dt + \int_0^T \|\nu^{1/2} \nabla \boldsymbol{e}_u\|_\Omega^2 dt \\
&\le K(K_\zeta) h^{2k} + \int_0^T \sum_e (\partial_t \theta_\zeta, e_\zeta)_{\Omega_e} dt - \int_0^T \sum_e (\boldsymbol{u}\,\zeta - \boldsymbol{U}\,\zeta_I, \nabla e_\zeta)_{\Omega_e} dt \\
&\qquad + \int_0^T \sum_i \langle \zeta\,\boldsymbol{u} \cdot \boldsymbol{n}_i - \zeta_I\,\boldsymbol{U} \cdot \boldsymbol{n}_i, [e_\zeta] \rangle_{\gamma_i} dt + \int_0^T \langle \theta_\zeta\,\hat{\boldsymbol{u}} \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_{\mathrm{O}}} dt \\
&\qquad - \frac{1}{2}\int_0^T \langle (\hat{\boldsymbol{u}} - \Pi\boldsymbol{u}) \cdot \boldsymbol{n}, e_\zeta^2 \rangle_{\partial\Omega_{\mathrm{O}}} dt + \int_0^T (g\nabla\theta_\zeta, \boldsymbol{e}_u)_\Omega dt \\
&\qquad - \frac{1}{2}\int_0^T \sum_e (\nabla \cdot \boldsymbol{U}, e_\zeta^2)_{\Omega_e} dt + \int_0^T \sum_e (g\,e_\zeta, \nabla \cdot \boldsymbol{e}_u)_{\Omega_e} dt \\
&\equiv K(K_\zeta) h^{2k} + T_1 + \ldots + T_8.
\end{aligned} \quad (31)$$

We now bound each of the terms on the right hand side of this equation. The first term can be bounded as

$$
\begin{aligned}
T_1 &= \int_0^T \sum_e (\partial_t \theta_\zeta, e_\zeta)_{\Omega_e} dt \\
&\le \int_0^T \|\partial_t \theta_\zeta\|_\Omega \, \|e_\zeta\|_\Omega dt \\
&\le \frac{1}{2}\int_0^T \|\partial_t \theta_\zeta\|_\Omega^2 dt + \frac{1}{2}\int_0^T \|e_\zeta\|_\Omega^2 dt \\
&\le K(K_\zeta) h^{2k} + \frac{1}{2}\int_0^T \|e_\zeta\|_\Omega^2 dt. \qquad (32)
\end{aligned}
$$

Integrate T_2 by parts and combine with T_3 and T_4 to obtain

$$
\begin{aligned}
T_2 + T_3 + T_4 = &-\int_0^T \sum_e (\boldsymbol{u}\,\zeta - \boldsymbol{U}\,\zeta_I, \nabla e_\zeta)_{\Omega_e} dt \\
&+ \int_0^T \sum_i \langle \zeta\,\boldsymbol{u}\cdot\boldsymbol{n}_i - \zeta_I\,\boldsymbol{U}\cdot\boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} dt + \int_0^T \langle \theta_\zeta \hat{\boldsymbol{u}}\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\mathrm{O}}} dt \\
= &\int_0^T \sum_e (\nabla\cdot(\boldsymbol{u}\,\zeta - \boldsymbol{U}\,\zeta_I), e_\zeta)_{\Omega_e} dt - \int_0^T \langle \hat{\zeta}\,\hat{\boldsymbol{u}}\cdot\boldsymbol{n} - \zeta_I\,\Pi\boldsymbol{u}\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\mathrm{I}}} dt \\
&- \int_0^T \langle \zeta_I(\hat{\boldsymbol{u}} - \Pi\boldsymbol{u})\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\mathrm{O}}} dt. \qquad (33)
\end{aligned}
$$

Bound the former term as

$$
\begin{aligned}
&\int_0^T \sum_e (\nabla\cdot(\boldsymbol{u}\,\zeta - \boldsymbol{U}\,\zeta_I), e_\zeta)_{\Omega_e} dt \\
&\quad= \int_0^T \sum_e ((\nabla\cdot\boldsymbol{u})\,\theta_\zeta, e_\zeta)_{\Omega_e} dt + \int_0^T \sum_e (\boldsymbol{u}\cdot\nabla\theta_\zeta, e_\zeta)_{\Omega_e} dt \\
&\quad\quad+ \int_0^T \sum_e (\zeta_I\,\nabla\cdot(\boldsymbol{\theta}_u - \boldsymbol{e}_u), e_\zeta)_{\Omega_e} dt + \int_0^T \sum_e (\nabla\zeta_I\cdot(\boldsymbol{\theta}_u - \boldsymbol{e}_u), e_\zeta)_{\Omega_e} dt \\
&\quad\le K(K_{\boldsymbol{u}}^*) \int_0^T \|\theta_\zeta\|_{H^1(\Omega)}^2 dt + K(K_\zeta^*) \int_0^T \|\boldsymbol{\theta}_u\|_{H^1(\Omega)}^2 dt \\
&\quad\quad+ K(K_\zeta^*, \nu) \int_0^T \|\boldsymbol{e}_u\|_\Omega^2 dt + K(K_\zeta^*, \nu) \int_0^T \|e_\zeta\|_\Omega^2 dt \\
&\quad\quad+ \frac{1}{8}\int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u\|_\Omega^2 dt \\
&\quad\le K(\nu, K_{\boldsymbol{u}}, K_\zeta, K_{\boldsymbol{u}}^*, K_\zeta^*) h^{2k} + K(K_\zeta^*, \nu) \int_0^T [\|\boldsymbol{e}_u\|_\Omega^2 + \|e_\zeta\|_\Omega^2] dt \\
&\quad\quad+ \frac{1}{8}\int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u\|_\Omega^2 dt. \qquad (34)
\end{aligned}
$$

and the boundary terms as

$$
\begin{aligned}
-\int_0^T \langle \hat{\zeta}\hat{\boldsymbol{u}}\cdot\boldsymbol{n} - \zeta_I\,\Pi\boldsymbol{u}\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\rm I}}dt - \int_0^T \langle \zeta_I(\hat{\boldsymbol{u}}-\Pi\boldsymbol{u})\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\rm O}}dt& \\
= -\int_0^T \langle \theta_\zeta\,\hat{\boldsymbol{u}}\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\rm I}}dt - \int_0^T \langle \zeta_I\,(\hat{\boldsymbol{u}}-\Pi\boldsymbol{u})\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega}dt& \\
\le K(K^*_{\boldsymbol{u}})\int_0^T h^{-1/2}\|\theta_\zeta\|_{\partial\Omega_{\rm I}}\, h^{1/2}\|e_\zeta\|_{\partial\Omega_{\rm I}}dt& \\
+K(K^*_\zeta)\int_0^T h^{-1/2}\|(\hat{\boldsymbol{u}}-\Pi\boldsymbol{u})\cdot\boldsymbol{n}\|_{\partial\Omega}\, h^{1/2}\|e_\zeta\|_{\partial\Omega}dt& \\
\le K(K^*_{\boldsymbol{u}},K^t)\int_0^T h^{-1}\|\theta_\zeta\|_\Omega\|\theta_\zeta\|_{H^1(\Omega)}dt + K(K^*_\zeta,K^t)\int_0^T h^{-1}\|\boldsymbol{\theta}_u\|_\Omega\|\boldsymbol{\theta}_u\|_{H^1(\Omega)}dt& \\
+K(K^t,K^i)\int_0^T \|e_\zeta\|^2_\Omega dt& \\
\le K(K^*_{\boldsymbol{u}},K^*_\zeta,K^t,K_{\boldsymbol{u}},K_\zeta,\nu)h^{2k} + K(K^t,K^i)\int_0^T \|e_\zeta\|^2_\Omega dt.& \qquad (35)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
T_5 &= -\frac{1}{2}\int_0^T \langle (\hat{\boldsymbol{u}}-\Pi\boldsymbol{u})\cdot\boldsymbol{n}, e^2_\zeta\rangle_{\partial\Omega_{\rm O}}dt \\
&\le \frac{1}{2}||e_\zeta||_{L^\infty(0,T;L^\infty(\Omega))}\int_0^T ||(\hat{\boldsymbol{u}}-\Pi\boldsymbol{u})\cdot\boldsymbol{n}||_{\partial\Omega_{\rm O}}||e_\zeta||_{\partial\Omega_{\rm O}}dt \\
&\le K(K^*_Z,K^*_\zeta,K^t,K_{\boldsymbol{u}},\nu)h^{2k} + K(K^t,K^i)\int_0^T \|e_\zeta\|^2_\Omega dt. \qquad (36)
\end{aligned}
$$

Integrate T_6 by parts to obtain

$$
\begin{aligned}
T_6 &= \int_0^T (g\nabla\theta_\zeta, \boldsymbol{e}_u)_\Omega dt \\
&= -\int_0^T (g\theta_\zeta, \nabla\cdot\boldsymbol{e}_u)_\Omega dt \\
&\le K(g,\nu,K_{\boldsymbol{u}})h^{2k} + \frac{1}{8}\int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u\|^2_\Omega dt. \qquad (37)
\end{aligned}
$$

We bound T_7 as

$$
\begin{aligned}
T_7 &= \frac{1}{2}\int_0^T \sum_e (-\nabla\cdot\boldsymbol{U}, e^2_\zeta)_{\Omega_e}dt \\
&= -\frac{1}{2}\int_0^T \sum_e (\nabla\cdot\boldsymbol{e}_u, e^2_\zeta)_\Omega dt + \frac{1}{2}\int_0^T \sum_e (\nabla\cdot\boldsymbol{\theta}_u, e^2_\zeta)_\Omega dt - \frac{1}{2}\int_0^T \sum_e (\nabla\cdot\boldsymbol{u}, e^2_\zeta)_\Omega dt \\
&\le K(K^*_\zeta,K^*_Z,K^*_{\boldsymbol{u}},\nu)\int_0^T ||e_\zeta||^2_\Omega dt + K(\nu,K_{\boldsymbol{u}},K^*_\zeta,K^*_Z)h^{2k} + \\
&\quad \frac{1}{8}\int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u||^2 dt. \qquad (38)
\end{aligned}
$$

Finally, T_8 can be bounded as

$$\begin{aligned} T_8 &= \int_0^T \sum_e (g\, e_\zeta, \nabla \cdot \boldsymbol{e}_u)_{\Omega_e} dt \\ &\leq K(g,\nu) \int_0^T \|e_\zeta\|_\Omega^2 dt + \frac{1}{8} \int_0^T \|\nu^{1/2} \nabla \boldsymbol{e}_u\|_\Omega^2 dt. \end{aligned} \tag{39}$$

Combining each of these results into (31) yields

$$\begin{aligned} &\frac{1}{2}\|e_\zeta(T)\|_\Omega^2 + \frac{1}{2}\|\boldsymbol{e}_u(T)\|_\Omega^2 \\ &\qquad + \frac{1}{2}\int_0^T \sum_i \langle |\boldsymbol{U}\cdot\boldsymbol{n}_i|, [e_\zeta]^2\rangle_{\gamma_i} dt + \frac{1}{2}\int_0^T \langle |\hat{\boldsymbol{u}}\cdot\boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega_{\mathrm{O}}} dt \\ &\qquad + \frac{1}{2}\int_0^T \langle |\Pi\boldsymbol{u}\cdot\boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega_{\mathrm{I}}} dt + \int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u\|_\Omega^2 dt \\ &\leq K(K_{\boldsymbol{u}}, K_\zeta, K^t, K^i, K_\zeta^*, K_Z^*, K_{\boldsymbol{u}}^*, g, \nu) h^{2k} + \frac{1}{2}\int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u\|_\Omega^2 dt \\ &\qquad + K(K^t, K_{\boldsymbol{u}}^*, K_\zeta^*, K_Z^*, g, \nu)\int_0^T [\|e_\zeta\|_\Omega^2 + \|\boldsymbol{e}_u\|_\Omega^2] dt. \end{aligned} \tag{40}$$

We hide the term $\frac{1}{2}\int_0^T \|\nu^{1/2}\nabla\boldsymbol{e}_u\|_\Omega^2 dt$ in the left hand side and apply Gronwall's lemma to obtain (20).

We recall another inverse inequality, valid in two dimensions,

$$||e_\zeta(\cdot,t)||_{L^\infty(\Omega)} \leq Kh^{-1}||e_\zeta(\cdot,t)||_\Omega,$$

where K is independent of h. Assuming $k > 1$ and h is sufficiently small, we have

$$\begin{aligned} ||Z||_{L^\infty(0,T;L^\infty(\Omega))} &\leq ||\zeta_I||_{L^\infty(0,T;L^\infty(\Omega))} + ||e_\zeta||_{L^\infty(0,T;L^\infty(\Omega))} \\ &\leq K_\zeta^* + KK_1h^{k-1} \\ &<< 2K_\zeta^* \\ &\leq K_Z^*. \end{aligned}$$

Thus, we can remove the dependence of K_1 on K_Z^*. □

By the triangle inequality, we easily obtain

COROLLARY 3.3.

$$||(\zeta - Z)(T)|| + ||(\boldsymbol{u} - \boldsymbol{U})(T)|| \leq K_1 h^k. \tag{41}$$

4. The discontinuous Galerkin and NIPG formulation. To formulate an alternate weak form of the shallow water system, we will discretize the primitive continuity equation by a discontinuous Galerkin finite element method and the corresponding momentum equation by the NIPG discontinuous Galerkin method [33]. In this case, the inflow and outflow boundary regions are defined as $\partial\Omega_{\mathrm{I}} = \{x \in \partial\Omega : \boldsymbol{U}\cdot\boldsymbol{n} < 0\}$ and $\partial\Omega_{\mathrm{O}} = \{x \in \partial\Omega : \boldsymbol{U}\cdot\boldsymbol{n} \geq 0\}$, respectively, for $\partial\Omega = \partial\Omega_{\mathrm{I}} \cup \partial\Omega_{\mathrm{O}}$, where $\boldsymbol{U} \approx \boldsymbol{u}$ is given by the NIPG method. In this case, we will assume only that the partitions $\mathcal{T}_h$ are nondegenerate [7]. In addition, $\mathcal{T}_h$ may possibly be nonconforming;

that is, element boundaries need not align. However, we will assume that any for any element Ω_e, the number of neighbors of Ω_e is bounded independently of h. A neighbor $\Omega_{e'}$ is any element such that

$$\text{interior of } \partial\Omega_e \cap \partial\Omega_{e'} \neq \emptyset. \tag{42}$$

For such neighboring elements, we also assume a local quasi-uniformity; that is, there exists a positive constant $K^q < 1$ independent of Ω_e and $\Omega_{e'}$ such that

$$K^q \leq \frac{h_e}{h_{e'}} \leq (K^q)^{-1}. \tag{43}$$

Each edge γ_i in the mesh is connected to at most a fixed, finite number of elements Ω_e as $h \to 0$. Denote this set of elements by $\mathcal{T}_{\gamma_i}$ and let h_{γ_i} denote the maximal element diameter over this set. Edges on $\partial\Omega$ are of course connected to only one element.

Multiply equations (1) and (2) by arbitrary, smooth test functions $v \in H^1(\Omega_e)$ and $\boldsymbol{w} \in (H^1(\Omega_e))^2$ respectively, then integrate by parts over each element Ω_e to obtain:

$$(\partial_t\zeta, v)_{\Omega_e} - (\boldsymbol{u}\,\zeta, \nabla v)_{\Omega_e} + \langle \zeta\,\boldsymbol{u}\cdot\boldsymbol{n}_e, v\rangle_{\partial\Omega_e} = 0, \tag{44}$$

$$(\partial_t\boldsymbol{u}, \boldsymbol{w})_{\Omega_e} + (g\nabla\zeta, \boldsymbol{w})_{\Omega_e} + (\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{w})_{\Omega_e} - \langle \nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}_e, \boldsymbol{w}\rangle_{\partial\Omega_e} = (\boldsymbol{f}, \boldsymbol{w})_{\Omega_e}. \tag{45}$$

Summing over all elements Ω_e and noting that both the velocity and its normal flux are continuous, we obtain the weak formulation:

$$\sum_e(\partial_t\zeta, v)_{\Omega_e} - \sum_e(\boldsymbol{u}\,\zeta, \nabla v)_{\Omega_e} + \sum_i\langle \zeta\,\boldsymbol{u}\cdot\boldsymbol{n}_i, [v]\rangle_{\gamma_i} + \langle \zeta\,\hat{\boldsymbol{u}}\cdot\boldsymbol{n}, v\rangle_{\partial\Omega_{\mathrm{O}}} = -\langle \hat{\zeta}\,\hat{\boldsymbol{u}}\cdot\boldsymbol{n}, v\rangle_{\partial\Omega_{\mathrm{I}}} \tag{46}$$

$$\begin{aligned}\sum_e(\partial_t\boldsymbol{u}, \boldsymbol{w})_{\Omega_e} &+ \sum_e(g\nabla\zeta, \boldsymbol{w})_{\Omega_e} - \sum_i\langle g\,[\zeta], \overline{\boldsymbol{w}}\cdot\boldsymbol{n}_i\rangle_{\gamma_i} + \sum_e(\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{w})_{\Omega_e} \\ &- \sum_i\langle\overline{\nu\nabla\boldsymbol{u}}\cdot\boldsymbol{n}_i, [\boldsymbol{w}]\rangle_{\gamma_i} + \sum_i\langle\overline{\nu\nabla\boldsymbol{w}}\cdot\boldsymbol{n}_i, [\boldsymbol{u}]\rangle_{\gamma_i} - \langle\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\Omega} \\ &+ \langle\nu\nabla\boldsymbol{w}\cdot\boldsymbol{n}, \boldsymbol{u}-\hat{\boldsymbol{u}}\rangle_{\partial\Omega} = \sum_e(\boldsymbol{f}, \boldsymbol{w})_{\Omega_e}.\end{aligned} \tag{47}$$

Note that the stabilization terms $-\langle g\,[\zeta], \overline{\boldsymbol{w}}\cdot\boldsymbol{n}_i\rangle_{\gamma_i}$ and $\langle\overline{\nu\nabla\boldsymbol{w}}\cdot\boldsymbol{n}_i, [\boldsymbol{u}]\rangle_{\gamma_i}$ are zero, as well as the boundary term $\langle\nu\nabla\boldsymbol{w}\cdot\boldsymbol{n}, \boldsymbol{u}-\hat{\boldsymbol{u}}\rangle_{\partial\Omega}$.

On each Ω_e, we will use the finite dimensional approximating spaces $V_h \subset H^1(\Omega_e)$ and $\boldsymbol{W}_h \subset (H^1(\Omega_e))^2$ defined by:

$$\begin{aligned} V_h &= \{v : \Omega \to I\!R : v|_{\Omega_e} \in \mathcal{P}^{k_e^\zeta}(\Omega_e)\} \\ \boldsymbol{W}_h &= \{\boldsymbol{w} : \Omega \to I\!R : \boldsymbol{w}|_{\Omega_e} \in (\mathcal{P}^{k_e^u}(\Omega_e))^2\}, \end{aligned}$$

where $k_e^\zeta, k_e^u \geq 1$. Approximate $\zeta(\cdot,t)$ by $Z(\cdot,t) \in V_h$, and $\boldsymbol{u}(\cdot,t)$ by $\boldsymbol{U}(\cdot,t) \in \boldsymbol{W}_h$. Sum (7) over all elements Ω_e, and, as before, let the value of ζ across inner element boundaries be approximated by the upwind value $Z^\uparrow$ where

$$\zeta \approx Z^\uparrow = \begin{cases} Z^-, & \overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i > 0, \\ Z^+, & \overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i \leq 0, \end{cases} \qquad \text{on } \gamma_i.$$

At $t = 0$ define $Z(\cdot, 0) = Z_0 \in V_h$ and $\boldsymbol{U}(\cdot, 0) = \boldsymbol{U}_0 \in \boldsymbol{W}_h$ by

$$(Z_0 - \zeta_0, v)_\Omega = 0, \qquad \forall\, v \in V_h, \tag{48}$$
$$(\boldsymbol{U}_0 - \boldsymbol{u}_0, \boldsymbol{w})_\Omega = 0, \qquad \forall\, \boldsymbol{w} \in \boldsymbol{W}_h. \tag{49}$$

The discrete weak formulation is: for each $t > 0$, find $(Z, \boldsymbol{U}) \in V_h \times \boldsymbol{W}_h$ satisfying

$$\sum_e (\partial_t Z, v)_{\Omega_e} - \sum_e (\boldsymbol{U}\, Z, \nabla v)_{\Omega_e} + \sum_i \langle Z^{\uparrow}\, \overline{\boldsymbol{U}} \cdot \boldsymbol{n}_i, [v] \rangle_{\gamma_i} + \langle Z\, \boldsymbol{U} \cdot \boldsymbol{n}, v \rangle_{\partial\Omega_{\mathrm{O}}} + \langle \hat{\zeta}\, \boldsymbol{U} \cdot \boldsymbol{n}, v \rangle_{\partial\Omega_{\mathrm{I}}} = 0, \tag{50}$$

$$\begin{aligned} \sum_e (\partial_t \boldsymbol{U}, \boldsymbol{w})_{\Omega_e} &+ \sum_e (g \nabla Z, \boldsymbol{w})_{\Omega_e} - \sum_i \langle g\,[Z], \overline{\boldsymbol{w}} \cdot \boldsymbol{n}_i \rangle_{\gamma_i} + \sum_e (\nu \nabla \boldsymbol{U}, \nabla \boldsymbol{w})_{\Omega_e} \\ &- \sum_i \langle \overline{\nu \nabla \boldsymbol{U}} \cdot \boldsymbol{n}_i, [\boldsymbol{w}] \rangle_{\gamma_i} + \sum_i \langle \overline{\nu \nabla \boldsymbol{w}} \cdot \boldsymbol{n}_i, [\boldsymbol{U}] \rangle_{\gamma_i} - \langle \nu \nabla \boldsymbol{U} \cdot \boldsymbol{n}, \boldsymbol{w} \rangle_{\partial\Omega} \\ &+ \langle \nu \nabla \boldsymbol{w} \cdot \boldsymbol{n}, \boldsymbol{U} - \hat{\boldsymbol{u}} \rangle_{\partial\Omega} + \sum_i \langle \sigma [\boldsymbol{U}], [\boldsymbol{w}] \rangle_{\gamma_i} \\ &+ \langle \sigma (\boldsymbol{U} - \hat{\boldsymbol{u}}), \boldsymbol{w} \rangle_{\partial\Omega} = \sum_e (\boldsymbol{f}, \boldsymbol{w})_{\Omega_e}, \end{aligned} \tag{51}$$

where we have introduced the "interior penalty" term $\sum_i \langle \sigma[\boldsymbol{U}], [\boldsymbol{w}] \rangle_{\gamma_i}$ and the "boundary penalty" term $\langle \sigma(\boldsymbol{U} - \hat{\boldsymbol{u}}), \boldsymbol{w} \rangle_{\partial\Omega}$. Here $\sigma > 0$. We assume

$$\sigma|_{\gamma_i} = \mathcal{O}(h_{\gamma_i}^{-1}). \tag{52}$$

4.1. An *a priori* error estimate. Let $\pi\zeta \in V_h$ and $\pi\boldsymbol{u} \in \boldsymbol{W}_h$ be the L^2 projections of ζ and $\boldsymbol{u}$, respectively; i.e.,

$$(\pi\zeta - \zeta, v)_{\Omega_e} = 0, \quad v \in V_h,$$
$$(\pi\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{w})_{\Omega_e} = 0, \quad \boldsymbol{w} \in \boldsymbol{W}_h.$$

Define

$$e_\zeta = Z - \pi\zeta, \qquad \boldsymbol{e}_u = \boldsymbol{U} - \pi\boldsymbol{u},$$
$$\theta_\zeta = \zeta - \pi\zeta, \qquad \boldsymbol{\theta}_u = \boldsymbol{u} - \pi\boldsymbol{u},$$

We will assume constants $K_Z^* \geq 2K_\zeta^*$ and $K_{\boldsymbol{U}}^* \geq 2K_{\boldsymbol{u}}^*$ exist such that

$$||Z||_{L^\infty(0,T;L^\infty(\Omega))} \leq K_Z^*,$$
$$||\boldsymbol{U}||_{L^\infty(0,T;L^\infty(\Omega))} \leq K_{\boldsymbol{U}}^*,$$

where K_ζ^* and $K_{\boldsymbol{u}}^*$ are defined by

$$K_{\boldsymbol{u}}^* \equiv ||\boldsymbol{u}||_{L^\infty(0,T;W_1^\infty(\Omega))}, \tag{53}$$
$$K_\zeta^* \equiv ||\pi\zeta||_{L^\infty(0,T;W_1^\infty(\Omega))}. \tag{54}$$

We will also use a local version of Theorem (3.1). In particular, for $v \in V_h$,

$$||v||_{H^1(\Omega_e)} \leq K_e^i h_e^{-1} ||v||_{\Omega_e}, \tag{55}$$

where K_e^i is independent of h_e. A similar result holds for $\boldsymbol{w} \in \boldsymbol{W}_h$. In the arguments below, let $K^i = \max_e K_e^i$.

THEOREM 4.1. *For $\boldsymbol{u}$,ζ sufficiently smooth and positive penalty parameter σ satisfying (52), the scheme (50)-(51) satisfies the error estimate*

$$\||(e_\zeta, \boldsymbol{e}_u)\|| \le K_2 \left\{ \int_0^T \sum_e \left[h_e^{2k_e^\zeta} ||\zeta||^2_{H^{k_e^\zeta+1}(\Omega_e)} + h_e^{2k_e^u} ||\boldsymbol{u}||^2_{H^{k_e^u+1}(\Omega_e)} \right] dt \right\}^{1/2},$$

where

$$\begin{aligned}
2\||(e_\zeta, \boldsymbol{e}_u)\||^2 &= \|e_\zeta(T)\|_\Omega^2 + \|\boldsymbol{e}_u(T)\|_\Omega^2 \\
&\quad + \int_0^T \|\nu^{1/2}\nabla \boldsymbol{e}_u\|_\Omega^2 dt + \int_0^T \sum_i \langle |\overline{\boldsymbol{U}} \cdot \boldsymbol{n}_i|, [e_\zeta]^2\rangle_{\gamma_i} dt \\
&\quad + \int_0^T \langle |\boldsymbol{U}\cdot \boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega} dt + \int_0^T \sum_i \|\sigma^{1/2}[\boldsymbol{e}_u]\|_{\gamma_i}^2 dt + \int_0^T \|\sigma^{1/2}\boldsymbol{e}_u|_{\partial\Omega}^2 dt,
\end{aligned}$$

and K_2 is independent of h, but depends on g, ν, K^q, K^i, K^t, K_ζ^*, $K_{\boldsymbol{u}}^*$, $K_{\boldsymbol{U}}^*$ and K_Z^*. For $k_e^\zeta, k_e^u > 1$ and h_e sufficiently small, the dependence of K_2 on K_Z^* and $K_{\boldsymbol{U}}^*$ can be removed.

Proof. Subtract the weak formulation (46)-(47) from the corresponding discrete formulation (50)-(51), and integrate in time from 0 to T. Incorporate the projections $\pi \boldsymbol{u}$ and $\pi\zeta$ and take $(v, \boldsymbol{w}) = (e_\zeta, \boldsymbol{e}_u)$ to obtain

$$\begin{aligned}
&\int_0^T \sum_e (\partial_t e_\zeta, e_\zeta)_{\Omega_e} dt - \int_0^T \sum_e (\boldsymbol{U}\, e_\zeta, \nabla e_\zeta)_{\Omega_e} dt + \int_0^T \sum_i \langle e_\zeta^\uparrow \overline{\boldsymbol{U}} \cdot \boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} dt \\
&\quad + \int_0^T \langle e_\zeta\, \boldsymbol{U}\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\rm O}} dt + \int_0^T \langle \hat{\zeta}\, \boldsymbol{U}\cdot \boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\rm I}} dt + \int_0^T \sum_e (\partial_t \boldsymbol{e}_u, \boldsymbol{e}_u)_{\Omega_e} dt \\
&+ \int_0^T \sum_e (g\nabla e_\zeta, \boldsymbol{e}_u)_{\Omega_e} dt - \int_0^T \sum_i \langle g\,[e_\zeta], \overline{\boldsymbol{e}_u}\cdot \boldsymbol{n}_i\rangle_{\gamma_i} dt + \int_0^T \sum_e (\nu\nabla \boldsymbol{e}_u, \nabla \boldsymbol{e}_u)_{\Omega_e} dt \\
&\qquad + \int_0^T \sum_i \langle \sigma[\boldsymbol{e}_u], [\boldsymbol{e}_u]\rangle_{\gamma_i} dt + \int_0^T \langle \sigma \boldsymbol{e}_u, \boldsymbol{e}_u\rangle_{\partial\Omega} dt \\
&= -\int_0^T \sum_e (\boldsymbol{u}\,\zeta - \boldsymbol{U}\pi\zeta, \nabla e_\zeta)_{\Omega_e} dt + \int_0^T \sum_i \langle (\zeta\, u - \pi\zeta^\uparrow\, \overline{\boldsymbol{U}})\cdot \boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} dt \\
&\quad + \int_0^T \langle (\zeta\,\hat{\boldsymbol{u}} - \pi\zeta \boldsymbol{U})\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\rm O}} dt + \int_0^T \langle \hat{\zeta}\, \hat{\boldsymbol{u}}\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\rm I}} dt \\
&\quad + \int_0^T \sum_e (g\nabla\theta_\zeta, \boldsymbol{e}_u)_{\Omega_e} dt - \int_0^T \sum_i \langle g\,[\theta_\zeta], \overline{\boldsymbol{e}_u}\cdot\boldsymbol{n}_i\rangle_{\gamma_i} dt \\
&\quad - \int_0^T \sum_i \langle \overline{\nu\nabla\boldsymbol{\theta}_u}\cdot\boldsymbol{n}_i, [\boldsymbol{e}_u]\rangle_{\gamma_i} dt + \int_0^T \sum_i \langle \overline{\nu\nabla \boldsymbol{e}_u}\cdot \boldsymbol{n}_i, [\boldsymbol{\theta}_u]\rangle_{\gamma_i} dt \\
&\quad - \int_0^T \langle \nu\nabla\boldsymbol{\theta}_u\cdot\boldsymbol{n}, \boldsymbol{e}_u\rangle_{\partial\Omega} dt + \int_0^T \langle \sigma(\pi\boldsymbol{u} - \hat{\boldsymbol{u}}), \boldsymbol{e}_u\rangle_{\partial\Omega} dt \\
&+ \int_0^T \sum_e (\nu\nabla\boldsymbol{\theta}_u, \nabla\boldsymbol{e}_u)_{\Omega_e} dt + \int_0^T \sum_i \langle\sigma[\boldsymbol{\theta}_u], [\boldsymbol{e}_u]\rangle_{\gamma_i} dt + \int_0^T \sum_i \langle \sigma\boldsymbol{\theta}_u, \boldsymbol{e}_u\rangle_{\partial\Omega} dt. \qquad (56)
\end{aligned}$$

We first consider the left hand side of this equation. The first and sixth terms are rewritten per equation (25) in the previous proof, noting that $e_\zeta(0) = \boldsymbol{e}_u(0) = 0$. Integrate the second term by parts to obtain

$$-\int_0^T \sum_e (\boldsymbol{U}\, e_\zeta, \nabla e_\zeta)_{\Omega_e} dt = \frac{1}{2}\int_0^T \sum_e (\nabla\cdot\boldsymbol{U}, e_\zeta^2)_{\Omega_e} dt - \frac{1}{2}\int_0^T \sum_i \langle [e_\zeta^2\, \boldsymbol{U}\cdot\boldsymbol{n}_i], 1\rangle_{\gamma_i} dt - \frac{1}{2}\int_0^T \langle \boldsymbol{U}\cdot\boldsymbol{n}, e_\zeta^2\rangle_{\partial\Omega} dt. \quad (57)$$

The third term of the left hand side can be combined with the jump term above to obtain

$$\begin{aligned}
&\int_0^T \sum_i \langle e_\zeta^\uparrow \overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} dt - \frac{1}{2}\int_0^T \sum_i \langle [e_\zeta^2\, \boldsymbol{U}\cdot\boldsymbol{n}_i], 1\rangle_{\gamma_i} dt \\
&= \int_0^T \sum_i \langle (e_\zeta^\uparrow \overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i\, [e_\zeta] - \frac{1}{2}[e_\zeta^2]\, \overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i - \frac{1}{2}\overline{e_\zeta^2}\, [\boldsymbol{U}]\cdot\boldsymbol{n}_i), 1\rangle_{\gamma_i} dt \\
&= \int_0^T \sum_i \langle (e_\zeta^\uparrow - \overline{e_\zeta})\, \overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} - \frac{1}{2}\int_0^T \sum_i \langle \overline{e_\zeta^2}\, [\boldsymbol{U}]\cdot\boldsymbol{n}_i, 1\rangle_{\gamma_i} dt \\
&= \frac{1}{2}\int_0^T \sum_i \langle |\overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i|, [e_\zeta]^2\rangle_{\gamma_i} dt - \frac{1}{2}\int_0^T \sum_i \langle [\boldsymbol{U}]\cdot\boldsymbol{n}_i, \overline{e_\zeta^2}\rangle_{\gamma_i} dt,
\end{aligned} \quad (58)$$

where we have used the fact that $[a\, b] = \overline{a}\, [b] + [a]\, \overline{b}$ and that $\frac{1}{2}[a^2] = [a]\, \overline{a}$. Moreover, the boundary term in (57) can be combined with the fourth term to obtain

$$-\frac{1}{2}\int_0^T \langle \boldsymbol{U}\cdot\boldsymbol{n}, e_\zeta^2\rangle_{\partial\Omega} dt + \int_0^T \langle \boldsymbol{U}\cdot\boldsymbol{n}, e_\zeta^2\rangle_{\partial\Omega_O} dt = \frac{1}{2}\int_0^T \langle |\boldsymbol{U}\cdot\boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega} dt. \quad (59)$$

Integrate the seventh term by parts and combine with the eighth term to obtain

$$\begin{aligned}
&\int_0^T \sum_e (g\nabla e_\zeta, \boldsymbol{e}_u)_{\Omega_e} dt - \int_0^T \sum_i \langle g\, [e_\zeta], \overline{\boldsymbol{e}_u}\cdot\boldsymbol{n}_i\rangle_{\gamma_i} dt \\
&= -\int_0^T \sum_e (g\, e_\zeta, \nabla\cdot\boldsymbol{e}_u)_{\Omega_e} dt + \int_0^T \sum_i \langle g\, \overline{e_\zeta}, [\boldsymbol{e}_u]\cdot\boldsymbol{n}_i\rangle_{\gamma_i} dt + \int_0^T \langle g\, e_\zeta, \boldsymbol{e}_u\cdot\boldsymbol{n}\rangle_{\partial\Omega} dt.
\end{aligned} \quad (60)$$

The remaining penalty terms can be written as

$$\int_0^T \sum_i \langle \sigma[\boldsymbol{e}_u], [\boldsymbol{e}_u]\rangle_{\gamma_i} dt + \int_0^T \langle \sigma\boldsymbol{e}_u, \boldsymbol{e}_u\rangle_{\partial\Omega} dt = \int_0^T \sum_i \|\sigma^{1/2}[\boldsymbol{e}_u]\|_{\gamma_i}^2 dt + \int_0^T \|\sigma^{1/2}\boldsymbol{e}_u\|_{\partial\Omega}^2 dt. \quad (61)$$

Combining each of these results into equation (56) yields

$$
\begin{aligned}
&\frac{1}{2}\|e_\zeta(T)\|_\Omega^2 + \frac{1}{2}\|\boldsymbol{e}_u(T)\|_\Omega^2 \\
&\quad + \int_0^T \|\nu^{1/2}\nabla \boldsymbol{e}_u\|_\Omega^2 dt + \frac{1}{2}\int_0^T \sum_i \langle|\overline{\boldsymbol{U}}\cdot \boldsymbol{n}_i|, [e_\zeta]^2\rangle_{\gamma_i} dt + \frac{1}{2}\int_0^T \langle|\boldsymbol{U}\cdot\boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega} dt \\
&\qquad + \int_0^T \sum_i \|\sigma^{1/2}[\boldsymbol{e}_u]\|_{\gamma_i}^2 dt + \int_0^T \|\sigma^{1/2}\boldsymbol{e}_u\|_{\partial\Omega}^2 dt \\
&= -\int_0^T \sum_e (\boldsymbol{u}\,\zeta - \boldsymbol{U}\pi\zeta, \nabla e_\zeta)_{\Omega_e} dt \\
&\quad + \int_0^T \sum_i \langle(\zeta\,\boldsymbol{u} - \pi\zeta^\uparrow \overline{\boldsymbol{U}})\cdot\boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} dt + \int_0^T \langle(\zeta\,\hat{\boldsymbol{u}} - \pi\zeta\boldsymbol{U})\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\mathrm{O}}} dt \\
&+ \int_0^T \langle\hat{\zeta}(\hat{\boldsymbol{u}} - \boldsymbol{U})\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega_{\mathrm{I}}} dt + \int_0^T \sum_e (g\nabla\theta_\zeta, \boldsymbol{e}_u)_{\Omega_e} dt - \int_0^T \sum_i \langle g\,[\theta_\zeta], \overline{\boldsymbol{e}_u}\cdot\boldsymbol{n}_i\rangle_{\gamma_i} dt \\
&- \int_0^T \sum_i \langle\overline{\nu\nabla\boldsymbol{\theta}_u}\cdot\boldsymbol{n}_i, [\boldsymbol{e}_u]\rangle_{\gamma_i} dt + \int_0^T \sum_i \langle\overline{\nu\nabla\boldsymbol{e}_u}\cdot\boldsymbol{n}_i, [\boldsymbol{\theta}_u]\rangle_{\gamma_i} dt - \int_0^T \langle\nu\nabla\boldsymbol{\theta}_u\cdot\boldsymbol{n}, \boldsymbol{e}_u\rangle_{\partial\Omega} dt \\
&\qquad - \frac{1}{2}\int_0^T \sum_e (\nabla\cdot\boldsymbol{U}, e_\zeta^2)_{\Omega_e} dt + \frac{1}{2}\int_0^T \sum_i \langle[\boldsymbol{U}]\cdot\boldsymbol{n}_i, \overline{e_\zeta^2}\rangle_{\gamma_i} dt \\
&\quad + \int_0^T \sum_e (g\,e_\zeta, \nabla\cdot\boldsymbol{e}_u)_{\Omega_e} dt - \int_0^T \sum_i \langle g\,\overline{e_\zeta}, [\boldsymbol{e}_u]\cdot\boldsymbol{n}_i\rangle_{\gamma_i} dt - \int_0^T \langle g\,e_\zeta, \boldsymbol{e}_u\cdot\boldsymbol{n}\rangle_{\partial\Omega} dt \\
&\quad + \int_0^T \sum_e (\nu\nabla\boldsymbol{\theta}_u, \nabla\boldsymbol{e}_u)_{\Omega_e} dt + \int_0^T \sum_i \langle\sigma[\boldsymbol{\theta}_u], [\boldsymbol{e}_u]\rangle_{\gamma_i} dt + \int_0^T \langle\sigma\boldsymbol{\theta}_u, \boldsymbol{e}_u\rangle_{\partial\Omega} dt \\
&= T_1 + \ldots + T_{17}. \qquad (62)
\end{aligned}
$$

We now bound terms T_1 through T_{17}.

T_1 is integrated by parts, yielding

$$
\begin{aligned}
T_1 &= -\int_0^T \sum_e (\boldsymbol{u}\,\zeta - \boldsymbol{U}\pi\zeta, \nabla e_\zeta)_{\Omega_e} dt \\
&= \int_0^T \sum_e (\nabla\cdot(\boldsymbol{u}\,\zeta - \boldsymbol{U}\pi\zeta), e_\zeta)_{\Omega_e} dt \\
&\quad - \int_0^T \sum_i \langle[e_\zeta(\zeta\,\boldsymbol{u} - \pi\zeta\,\boldsymbol{U})\cdot\boldsymbol{n}_i], 1\rangle_{\gamma_i} dt - \int_0^T \langle(\zeta\,\boldsymbol{u} - \pi\zeta\,\boldsymbol{U})\cdot\boldsymbol{n}, e_\zeta\rangle_{\partial\Omega} dt. \quad (63)
\end{aligned}
$$

Bound the first term above similar to equation (34), giving

$$\int_0^T \sum_e (\nabla\cdot(\boldsymbol{u}\,\zeta - \boldsymbol{U}\pi\zeta), e_\zeta)_{\Omega_e} dt \le K(K^*_{\boldsymbol{u}}) \int_0^T \sum_e \|\theta_\zeta\|^2_{H^1(\Omega_e)} dt$$
$$+K(K^*_\zeta) \int_0^T \sum_e \|\boldsymbol{\theta}_u\|^2_{H^1(\Omega_e)} dt$$
$$+K(K^*_\zeta,\nu) \int_0^T \|\boldsymbol{e}_u\|^2_\Omega dt + K(K^*_\zeta,\nu)\int_0^T \|e_\zeta\|^2_\Omega dt$$
$$+\frac{1}{8}\int_0^T \|\nu^{1/2}\nabla \boldsymbol{e}_u\|^2_\Omega dt. \tag{64}$$

The second term in (63) can be combined with T_2 to yield

$$T_2 - \int_0^T \sum_i \langle [e_\zeta\,(\zeta\,\boldsymbol{u} - \pi\zeta\,\boldsymbol{U})\cdot \boldsymbol{n}_i], 1\rangle_{\gamma_i} dt$$
$$= \int_0^T \sum_i \langle (\overline{\zeta \boldsymbol{u} - \pi\zeta^\uparrow \boldsymbol{U}})\cdot \boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} dt - \int_0^T \sum_i \langle [e_\zeta\,(\zeta\,\boldsymbol{u} - \pi\zeta\,\boldsymbol{U})\cdot \boldsymbol{n}_i], 1\rangle_{\gamma_i} dt$$
$$= \int_0^T \sum_i [\langle (\theta_\zeta^\uparrow - \overline{\theta_\zeta})\overline{\boldsymbol{U}}\cdot \boldsymbol{n}_i - \frac{1}{4}[\theta_\zeta][\boldsymbol{U}]\cdot\boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i}$$
$$+ \int_0^T \sum_i \langle \overline{\pi\zeta}([\boldsymbol{e}_u] - [\boldsymbol{\theta}_u])\cdot \boldsymbol{n}_i - [\theta_\zeta]\overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i, \overline{e_\zeta}\rangle_{\gamma_i} dt$$
$$\equiv T_{2,a} + T_{2,b} + T_{2,c} + T_{2,d}, \tag{65}$$

where we have again used $[a\,b] = \overline{a}\,[b] + [a]\,\overline{b}$ and $\overline{ab} = \overline{a}\,\overline{b} + \frac{1}{4}[a][b]$.

Consider

$$T_{2,a} = \int_0^T \sum_i \langle (\theta_\zeta^\uparrow - \overline{\theta_\zeta})\overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} dt. \tag{66}$$

We will use an argument here which we will emulate several times below in various forms:

$$\sum_i \langle \theta_\zeta^\uparrow \overline{\boldsymbol{U}}\cdot\boldsymbol{n}_i, [e_\zeta]\rangle_{\gamma_i} \le K(K^*_{\boldsymbol{U}}) \sum_i ||\theta_\zeta^\uparrow||_{\gamma_i} ||[e_\zeta]||_{\gamma_i}$$
$$\le K(K^*_{\boldsymbol{U}}, K^t) \sum_e ||\theta_\zeta||^{1/2}_{\Omega_e} ||\theta_\zeta||^{1/2}_{H^1(\Omega_e)} ||e_\zeta||^{1/2}_{\Omega_e} ||e_\zeta||^{1/2}_{H^1(\Omega_e)}$$
$$\le K(K^*_{\boldsymbol{U}}, K^t) \sum_e h_e^{-1} ||\theta_\zeta||_{\Omega_e} ||\theta_\zeta||_{H^1(\Omega_e)} + \sum_e h_e ||e_\zeta||_{\Omega_e} ||e_\zeta||_{H^1(\Omega_e)}$$
$$\le K(K^*_{\boldsymbol{U}}, K^t) \sum_e h_e^{-1} ||\theta_\zeta||_{\Omega_e} ||\theta_\zeta||_{H^1(\Omega_e)} + K(K^i) \sum_e ||e_\zeta||^2_{\Omega_e}. \tag{67}$$

Thus

$$T_{2,a} \le K(K^*_{\boldsymbol{U}}) \int_0^T \sum_i \left(||\theta_\zeta^\uparrow||_{\gamma_i} + ||\overline{\theta_\zeta}||_{\gamma_i}\right) ||[e_\zeta]||_{\gamma_i} dt$$
$$\le K(K^*_{\boldsymbol{U}}, K^t) \int_0^T \sum_e h_e^{-1} ||\theta_\zeta||_{\Omega_e} ||\theta_\zeta||_{H^1(\Omega_e)} + K(K^i) \int_0^T ||e_\zeta||^2_\Omega dt. \tag{68}$$

Terms $T_{2,b}$ and $T_{2,d}$ have similar bounds. Consider

$$T_{2,c} = \int_0^T \sum_i \langle \overline{\pi\zeta}([\boldsymbol{e}_u] - [\boldsymbol{\theta}_u]) \cdot \boldsymbol{n}_i, \overline{e_\zeta} \rangle_{\gamma_i} dt. \tag{69}$$

Here we will use another argument, repeated in various forms several times below. In particular,

$$\begin{aligned}
\int_0^T \sum_i \langle [\boldsymbol{e}_u], \overline{e_\zeta} \rangle_{\gamma_i} dt &= \int_0^T \sum_i \langle \sigma^{1/2}[\boldsymbol{e}_u], \sigma^{-1/2}\overline{e_\zeta} \rangle_{\gamma_i} dt \\
&\leq \epsilon \int_0^T \sum_i ||\sigma^{1/2}[\boldsymbol{e}_u]||^2_{\gamma_i} dt + K \int_0^T \sum_i ||\sigma^{-1/2}\overline{e_\zeta}||^2_{\gamma_i} dt \\
&\leq \epsilon \int_0^T \sum_i ||\sigma^{1/2}[\boldsymbol{e}_u]||^2_{\gamma_i} dt \\
&\quad + K(K^t) \int_0^T \sum_i \sigma^{-1}_{\gamma_i} \sum_{\Omega_e \in \mathcal{T}_{\gamma_i}} ||e_\zeta||_{\Omega_e} ||e_\zeta||_{H^1(\Omega_e)} dt \\
&\leq \epsilon \int_0^T \sum_i ||\sigma^{1/2}[\boldsymbol{e}_u]||^2_{\gamma_i} dt \\
&\quad + K(K^t, K^q) \int_0^T \sum_i \sum_{\Omega_e \in \mathcal{T}_{\gamma_i}} h_e ||e_\zeta||_{\Omega_e} ||e_\zeta||_{H^1(\Omega_e)} dt \\
&\leq \epsilon \int_0^T \sum_i ||\sigma^{1/2}[\boldsymbol{e}_u]||^2_{\gamma_i} dt \\
&\quad + K(K^t, K^q, K^i) \int_0^T \sum_e ||e_\zeta||^2_{\Omega} dt,
\end{aligned} \tag{70}$$

where we have used the trace inequality and inverse inequality (55), (52) and (43). Thus,

$$\begin{aligned}
T_{2,c} &= \int_0^T \sum_i \langle \overline{\pi\zeta}([\boldsymbol{e}_u] - [\boldsymbol{\theta}_u]) \cdot \boldsymbol{n}_i, \overline{e_\zeta} \rangle_{\gamma_i} dt \\
&= \int_0^T \sum_i \langle \sigma^{1/2}\overline{\pi\zeta}[\boldsymbol{e}_u] \cdot \boldsymbol{n}, \sigma^{-1/2}\overline{e_\zeta} \rangle_{\gamma_i} dt - \int_0^T \sum_i \langle \overline{\pi\zeta}[\boldsymbol{\theta}_u] \cdot \boldsymbol{n}_i, \overline{e_\zeta} \rangle_{\gamma_i} dt \\
&\leq \frac{1}{8} \int_0^T \sum_i ||\sigma^{1/2}[\boldsymbol{e}_u]||^2_{\gamma_i} dt + K(K^*_\zeta, K^t, K^i, K^q) \int_0^T ||e_\zeta||^2_{\Omega} dt \\
&\quad + K(K^t) \int_0^T \sum_e h_e^{-1} ||\boldsymbol{\theta}_u||_{\Omega_e} ||\boldsymbol{\theta}_u||_{H^1(\Omega_e)} dt.
\end{aligned} \tag{71}$$

The remaining boundary term in (63) can be combined with T_3 and T_4, yielding

$$
\begin{aligned}
T_3 + T_4 - \int_0^T & \langle (\zeta\, \boldsymbol{u} - \pi\zeta\, \boldsymbol{U}) \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega} dt \\
&= \int_0^T \langle (\zeta\, \hat{\boldsymbol{u}} - \pi\zeta \boldsymbol{U}) \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_{\mathrm{O}}} dt \\
&\quad \int_0^T \langle \hat{\zeta}(\hat{\boldsymbol{u}} - \boldsymbol{U}) \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_{\mathrm{I}}} dt - \int_0^T \langle (\zeta\, \boldsymbol{u} - \pi\zeta\, \boldsymbol{U}) \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega} dt \\
&= -\int_0^T \langle \theta_\zeta\, \boldsymbol{e}_u \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_{\mathrm{I}}} dt + \int_0^T \langle \theta_\zeta\, \boldsymbol{\theta}_u \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_{\mathrm{I}}} dt \\
&\quad - \int_0^T \langle \theta_\zeta\, \boldsymbol{u} \cdot \boldsymbol{n}, e_\zeta \rangle_{\partial\Omega_{\mathrm{I}}} dt \\
&\leq K(K_Z^*, K_\zeta^*) \int_0^T ||\boldsymbol{e}_u \cdot \boldsymbol{n}||_{\partial\Omega_{\mathrm{I}}} ||\theta_\zeta||_{\partial\Omega_{\mathrm{I}}} dt + K(K_\zeta^*) \int_0^T ||e_\zeta||_{\partial\Omega_{\mathrm{I}}} ||\boldsymbol{\theta}_u \cdot \boldsymbol{n}||_{\partial\Omega_{\mathrm{I}}} dt \\
&\quad + K(K_{\boldsymbol{u}}^*) \int_0^T ||\theta_\zeta||_{\partial\Omega_{\mathrm{I}}} ||e_\zeta||_{\partial\Omega_{\mathrm{I}}} dt \\
&\leq K(K^t, K^i) \int_0^T \left(||\boldsymbol{e}_u||_\Omega^2 + ||e_\zeta||_\Omega^2 \right) dt \\
&\quad + K(K^t, K_Z^*, K_\zeta^*) \int_0^T \sum_e h_e^{-1} ||\boldsymbol{\theta}_u||_{\Omega_e} ||\boldsymbol{\theta}_u||_{H^1(\Omega_e)} dt \\
&\quad + K(K_{\boldsymbol{u}}^*, K^t) \int_0^T \sum_e h_e^{-1} ||\theta_\zeta||_{\Omega_e} ||\theta_\zeta||_{H^1(\Omega_e)} dt. \qquad (72)
\end{aligned}
$$

Integrate T_5 by parts and combine with T_6 to obtain

$$
\begin{aligned}
T_5 + T_6 &= \int_0^T \sum_e (g \nabla \theta_\zeta, \boldsymbol{e}_u)_{\Omega_e} dt - \int_0^T \sum_i \langle g\, [\theta_\zeta], \overline{\boldsymbol{e}_u} \cdot \boldsymbol{n}_i \rangle_{\gamma_i} dt \\
&= -\int_0^T \sum_e (g\theta_\zeta, \nabla \cdot \boldsymbol{e}_u)_{\Omega_e} dt + \int_0^T \sum_i \langle g\, \overline{\theta_\zeta}, [\boldsymbol{e}_u] \cdot \boldsymbol{n}_i \rangle_{\gamma_i} dt \\
&\quad + \int_0^T \langle g\, \theta_\zeta, \boldsymbol{e}_u \cdot \boldsymbol{n} \rangle_{\partial\Omega} dt \\
&\leq K(g, \nu) \int_0^T \sum_e ||\theta_\zeta||_{\Omega_e}^2 dt + K(g) \int_0^T \sum_i ||\sigma^{-1/2} \overline{\theta_\zeta}||_{\gamma_i}^2 dt \\
&\quad + K(g) \int_0^T \sum_i ||\sigma^{-1/2} \theta_\zeta||_{\partial\Omega}^2 dt + \frac{1}{16} \int_0^T \sum_i ||\sigma^{1/2} [\boldsymbol{e}_u]||_{\gamma_i}^2 dt \\
&\quad + \frac{1}{8} \int_0^T ||\sigma^{1/2} \boldsymbol{e}_u||_{\partial\Omega}^2 dt + \frac{1}{8} \int_0^T ||\nu^{1/2} \nabla \boldsymbol{e}_u||_\Omega^2 dt \\
&\leq K(g, \nu, K^t, K^q) \int_0^T \sum_e \left[||\theta_\zeta||_{\Omega_e}^2 + h_e ||\theta_\zeta||_{\Omega_e} ||\theta_\zeta||_{H^1(\Omega_e)} \right] dt \\
&\quad + \frac{1}{16} \int_0^T \left\{ 2 \sum_i ||\sigma^{1/2} [\boldsymbol{e}_u]||_{\gamma_i}^2 + 2 ||\sigma^{1/2} \boldsymbol{e}_u||_{\partial\Omega}^2 + ||\nu^{1/2} \nabla \boldsymbol{e}_u||_\Omega^2 \right\} dt. \qquad (73)
\end{aligned}
$$

Continuing,

$$\begin{aligned}T_7 &= -\int_0^T \sum_i \langle \overline{\nu \nabla \boldsymbol{\theta}_u} \cdot \boldsymbol{n}_i, [\boldsymbol{e}_u] \rangle_{\gamma_i} dt \\ &\le K(\nu) \int_0^T \sum_i \|\sigma^{-1/2} \overline{\nabla \boldsymbol{\theta}_u} \cdot \boldsymbol{n}_i\|_{\gamma_i}^2 dt + \frac{1}{8} \int_0^T \sum_i \|\sigma^{1/2} [\boldsymbol{e}_u]\|_{\gamma_i}^2 dt \\ &\le K(\nu, K^t, K^q) \int_0^T \sum_e h_e ||\nabla \boldsymbol{\theta}_u||_{\Omega_e} ||\nabla \boldsymbol{\theta}_u||_{H^1(\Omega_e)} dt \\ &\quad + \frac{1}{16} \int_0^T \sum_i \|\sigma^{1/2} [\boldsymbol{e}_u]\|_{\gamma_i}^2 dt, \end{aligned} \tag{74}$$

and

$$\begin{aligned}T_8 &= \int_0^T \sum_i \langle \overline{\nu \nabla \boldsymbol{e}_u} \cdot \boldsymbol{n}_i, [\boldsymbol{\theta}_u] \rangle_{\gamma_i} dt \\ &\le \int_0^T \sum_i \|\overline{\nu^{1/2} \nabla \boldsymbol{e}_u} \cdot \boldsymbol{n}_i\|_{\gamma_i} \|\nu^{1/2} [\boldsymbol{\theta}_u]\|_{\gamma_i} dt \\ &\le \frac{1}{16} \int_0^T \sum_e \|\nu^{1/2} \nabla \boldsymbol{e}_u\|_{\Omega_e}^2 dt \\ &\quad + K(\nu, K^t, K^i) \int_0^T \sum_e h_e^{-1} ||\boldsymbol{\theta}_u||_{\Omega_e} ||\boldsymbol{\theta}_u||_{H^1(\Omega_e)} dt. \end{aligned} \tag{75}$$

$$\begin{aligned}T_9 &= -\int_0^T \langle \nu \nabla \boldsymbol{\theta}_u \cdot \boldsymbol{n}, \boldsymbol{e}_u \rangle_{\partial\Omega} dt \\ &\le K(\nu) \int_0^T \|\sigma^{-1/2} \nabla \boldsymbol{\theta}_u \cdot \boldsymbol{n}\|_{\partial\Omega}^2 dt + \frac{1}{8} \int_0^T \|\sigma^{1/2} \boldsymbol{e}_u\|_{\partial\Omega}^2 dt \\ &\le K(\nu, K^t, K^q) \int_0^T \sum_e h_e ||\nabla \boldsymbol{\theta}_u||_{\Omega_e} ||\nabla \boldsymbol{\theta}_u||_{H^1(\Omega_e)} dt + \frac{1}{8} \int_0^T \|\sigma^{1/2} \boldsymbol{e}_u\|_{\partial\Omega}^2 dt. \end{aligned} \tag{76}$$

Apply the steps in equation (38) to bound T_{10} as

$$\begin{aligned}T_{10} &= -\frac{1}{2} \int_0^T \sum_e (\nabla \cdot \boldsymbol{U}, e_\zeta^2)_{\Omega_e} dt \\ &= -\frac{1}{2} \int_0^T \sum_e (\nabla \cdot (\boldsymbol{e}_u - \boldsymbol{\theta}_u + \boldsymbol{u}), e_\zeta^2)_{\Omega_e} dt \\ &\le \frac{1}{16} \int_0^T ||\nu^{1/2} \nabla \boldsymbol{e}_u||_\Omega^2 dt + K(\nu, K_\zeta^*, K_Z^*, K_{\boldsymbol{u}}^*) \int_0^T ||e_\zeta||_\Omega^2 dt \\ &\quad + K(K_\zeta^*, K_Z^*) \int_0^T \sum_e ||\nabla \boldsymbol{\theta}_u||_{\Omega_e}^2 dt. \end{aligned} \tag{77}$$

Next,

$$
\begin{aligned}
T_{11} &= \frac{1}{2}\int_0^T \sum_i \langle [\boldsymbol{U}]\cdot\boldsymbol{n}_i, \overline{e_\zeta^2}\rangle_{\gamma_i} dt = \frac{1}{2}\int_0^T \sum_i \langle [\boldsymbol{e}_u]\cdot\boldsymbol{n}_i - [\boldsymbol{\theta}_u]\cdot\boldsymbol{n}_i, \overline{e_\zeta^2}\rangle_{\gamma_i} dt \\
&\le \frac{1}{2}\int_0^T \sum_i \|\sigma^{1/2}[\boldsymbol{e}_u]\cdot\boldsymbol{n}_i\|_{\gamma_i}\,\|\sigma^{-1/2}\overline{e_\zeta^2}\|_{\gamma_i} dt \\
&\quad +\frac{1}{2}\int_0^T \sum_i \|[\boldsymbol{\theta}_u]\cdot\boldsymbol{n}_i\|_{\gamma_i}\,\|\overline{e_\zeta^2}\|_{\gamma_i} dt \\
&\le \frac{1}{16}\int_0^T \sum_i \|\sigma^{1/2}[\boldsymbol{e}_u]\|_{\gamma_i}^2 dt \\
&\quad +K(K_\zeta^*, K_Z^*, K^t, K^i)\int_0^T \|e_\zeta\|_\Omega^2 dt \\
&\quad +K(K^t)\int_0^T \sum_e h_e^{-1}||\boldsymbol{\theta}_u||_{\Omega_e}||\boldsymbol{\theta}_u||_{H^1(\Omega_e)} dt. \qquad (78)
\end{aligned}
$$

T_{12} is easily bounded as

$$
\begin{aligned}
T_{12} &= \int_0^T \sum_e (g e_\zeta, \nabla\cdot\boldsymbol{e}_u)_{\Omega_e} dt \\
&\le K(g,\nu)\int_0^T \sum_e \|e_\zeta\|_{\Omega_e}^2 dt + \frac{1}{8}\int_0^T \sum_e \|\nu^{1/2}\nabla\boldsymbol{e}_u\|_{\Omega_e}^2 dt, \qquad (79)
\end{aligned}
$$

while the bound on T_{13} and T_{14} is

$$
\begin{aligned}
T_{13}+T_{14} &= -\int_0^T \sum_i \langle g\,\overline{e_\zeta}, [\boldsymbol{e}_u]\cdot\boldsymbol{n}_i\rangle_{\gamma_i} dt - \int_0^T \langle g e_\zeta, \boldsymbol{e}_u\cdot\boldsymbol{n}\rangle_{\partial\Omega} dt \\
&\le \frac{1}{16}\int_0^T \sum_i \|\sigma^{1/2}[\boldsymbol{e}_u]\|_{\gamma_i}^2 dt + K(K^t, K^i, K^q)\int_0^T \|e_\zeta\|_\Omega^2 dt \\
&\quad +\frac{1}{8}\int_0^T \sum_i \|\sigma^{1/2}\boldsymbol{e}_u\|_{\partial\Omega}^2 dt. \qquad (80)
\end{aligned}
$$

$$
\begin{aligned}
T_{15} &= \int_0^T \sum_e (\nu\nabla\boldsymbol{\theta}_u, \nabla\boldsymbol{e}_u)_{\Omega_e} dt \\
&\le K(\nu)\int_0^T \sum_e ||\nabla\boldsymbol{\theta}_u||_{\Omega_e}^2 dt + \frac{1}{16}\int_0^T ||\nu^{1/2}\nabla\boldsymbol{e}_u||_\Omega^2 dt. \qquad (81)
\end{aligned}
$$

Finally

$$
\begin{aligned}
T_{16}+T_{17} &= \int_0^T \left[\sum_i \langle\sigma[\boldsymbol{\theta}_u], [\boldsymbol{e}_u]\rangle_{\gamma_i} + \langle\sigma\boldsymbol{\theta}_u, \boldsymbol{e}_u\rangle_{\partial\Omega}\right] dt \\
&\le K(K^t, K^q)\int_0^T \sum_e h_e^{-1}||\boldsymbol{\theta}_u||_{\Omega_e}||\boldsymbol{\theta}_u||_{H^1(\Omega_e)} dt \\
&\quad +\frac{1}{16}\int_0^T \left[\sum_i ||\sigma^{1/2}[\boldsymbol{e}_u]||_{\gamma_i}^2 + ||\sigma^{1/2}\boldsymbol{e}_u||_{\partial\Omega}^2\right] dt. \qquad (82)
\end{aligned}
$$

Combining each of these results into (62) and hiding terms yields

$$\begin{aligned}
&\|e_\zeta(T)\|_\Omega^2 + \|\boldsymbol{e}_u(T)\|_\Omega^2 \\
&\quad + \int_0^T \|\nu^{1/2}\nabla \boldsymbol{e}_u\|_\Omega^2 dt + \int_0^T \sum_i \langle |\overline{\boldsymbol{U}}\cdot \boldsymbol{n}_i|, [e_\zeta]^2\rangle_{\gamma_i} dt + \int_0^T \langle |\boldsymbol{U}\cdot\boldsymbol{n}|, e_\zeta^2\rangle_{\partial\Omega} dt \\
&\qquad + \int_0^T \sum_i \|\sigma^{1/2}[\boldsymbol{e}_u]\cdot \boldsymbol{n}_i\|_{\gamma_i}^2 dt + \int_0^T \|\sigma^{1/2}\boldsymbol{e}_u\cdot\boldsymbol{n}\|_{\partial\Omega}^2 dt \\
&\qquad\qquad \le K_3 \int_0^T \left[\|e_\zeta\|_\Omega^2 + \|\boldsymbol{e}_u\|_\Omega^2\right] dt. \\
&\qquad + K_3 \left\{ \int_0^T \sum_e \left[h_e^{-1}||\theta_\zeta||_{\Omega_e}||\theta_\zeta||_{H^1(\Omega_e)} + h_e^{-1}||\boldsymbol{\theta}_u||_{\Omega_e}||\boldsymbol{\theta}_u||_{H^1(\Omega_e)} \right.\right. \\
&\qquad\quad \left.\left. + h_e||\nabla\boldsymbol{\theta}_u||_{\Omega_e}||\nabla\boldsymbol{\theta}_u||_{H^1(\Omega_e)} + ||\theta_\zeta||_{H^1(\Omega_e)}^2 + ||\boldsymbol{\theta}_u||_{H^1(\Omega_e)}^2 \right] dt \right\}, \quad (83)
\end{aligned}$$

where $K_3 = K(K^q, K^t, K^i, K_{\boldsymbol{u}}^*, K_\zeta^*, K_Z^*, K_{\boldsymbol{U}}^*, g, \nu)$. Assuming sufficient smoothness of ζ and $\boldsymbol{u}$, local error estimates for the L^2 projections $\pi\zeta$ and $\pi\boldsymbol{u}$ give

$$\begin{aligned}
h_e^{-1}||\theta_\zeta||_{\Omega_e} + ||\theta_\zeta||_{H^1(\Omega_e)} &\le K_e(h_e)^{k_e^\zeta}||\zeta||_{H^{k_e^\zeta+1}(\Omega_e)} \\
h_e^{-2}||\boldsymbol{\theta}_u||_{\Omega_e} + h_e^{-1}||\boldsymbol{\theta}_u||_{H^1(\Omega_e)} + ||\nabla\boldsymbol{\theta}_u||_{H^1(\Omega_e)} &\le K_e(h_e)^{k_e^u-1}||\boldsymbol{u}||_{H^{k_e^u+1}(\Omega_e)},
\end{aligned}$$

where K_e is a constant independent of h_e. Substituting these estimates into (83) and applying Gronwall's Lemma, we obtain (56).

The dependence of K_2 on K_Z^* and $K_{\boldsymbol{U}}^*$ can be removed for h_e sufficiently small and $k_e^\zeta, k_e^u \ge 1$ as in the proof of Theorem 1. □

5. Numerical Results. In this section we present some one-dimensional numerical results for the schemes discussed above. The first test case examines the order of convergence of both methods, while the second test case comapres each method's ability to model tidal fluctuations. The DG/conforming Galerkin finite element discretization with linear basis functions and the DG/NIPG method with $\mathcal{P}^1$ Legendre polynomials are implemented on elements of equal size. Time discretization is by Euler's method with a sufficiently small timestep so that temporal error is negligible compared to spatial error.

5.1. Test Case 1. We consider the test problem

$$\begin{aligned}
\partial_t\zeta + \partial_x(u\zeta) &= f \qquad (x,t)\in[0,\pi/4],\ t>0, \\
\partial_t\boldsymbol{u} + g\nabla\zeta - \nu\partial_{xx}u &= f \qquad (x,t)\in[0,\pi/4],\ t>0
\end{aligned}$$

for $g = 9.81$, $\nu = .01$, and impose boundary conditions such that the exact solution is

$$\begin{aligned}
\zeta &= \cos(x-t), \\
u &= \sin(x+t).
\end{aligned}$$

Tables 1 and 2 contain the L^2 errors measured at time $T = .5$ and corresponding convergence rates for both methods applied to this problem. The NIPG method incorporates a penalty parameter $\sigma = \mathcal{O}(1/h)$. For each discretization, "optimal" order $\mathcal{O}(h^{p+1})$ (for this case $p = 1$) accuracy was observed which is better than what we have proved above. However, this optimal numerical convergence is typical of the discontinuous Galerkin methods applied to convection-diffustion type problems.

Table 1. Convergence results for the discontinuous and continuous Galerkin formulation.

N	*error in* ζ	*rate*	*error in* u	*rate*
32	.00005642		.00003271	
64	.00001403	2.01	.00000821	1.99
128	.00000353	1.99	.00000208	1.98
256	.00000090	1.97	.00000053	1.97
512	.00000023	1.97	.00000013	2.03

Table 2. Convergence results for the discontinuous Galerkin and NIPG formulation.

N	*error in* ζ	*rate*	*error in* u	*rate*
32	.00005493		.00002046	
64	.00001378	2.00	.00000547	1.90
128	.00000348	1.99	.00000145	1.92
256	.00000088	1.98	.00000038	1.93
512	.00000022	2.00	.00000009	2.01

5.2. Test Case 2. In order to compare the methods' ability to mimic tidal fluctuations, we consider the test problem

$$\begin{aligned} \partial_t \zeta + \partial_x (uz) &= 0 \qquad (x,t) \in [0,10000],\ t > 0, \\ \partial_t \boldsymbol{u} + g\nabla\zeta - \nu\partial_{xx}u + \tau u &= 0 \qquad (x,t) \in [0,10000],\ t > 0, \end{aligned}$$

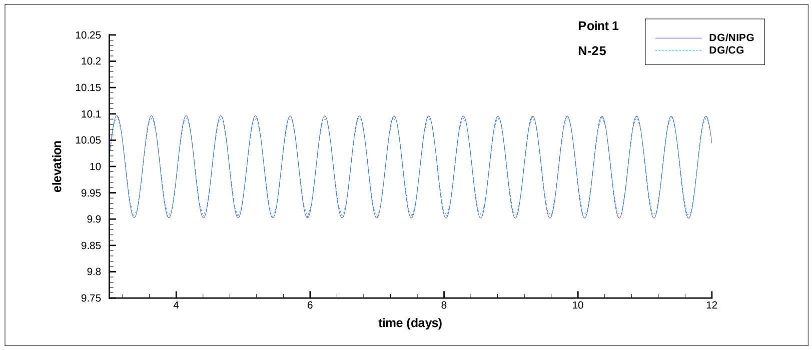
where $z = \zeta + h_b$ for mean sea level $h_b(x) = 10$. Coefficients are defined as $g = 9.81$, $\tau = .01$, and $\nu = 25$. Our initial and boundary conditions are of the form

$$\begin{aligned} \zeta(x,t_0) &= 0 \qquad \text{at } t_0 = 0, \\ u(x,t_0) &= 0 \qquad \text{at } t_0 = 0, \\ \partial_x u(x_L,t) &= 0 \qquad \text{at } x_L = 0, \\ u(x_R,t) &= 0 \qquad \text{at } x_R = 10000, \\ \zeta(x_L,t) &= .1\ \cos(t\,\alpha) \qquad \text{at } x_L = 0, \end{aligned}$$

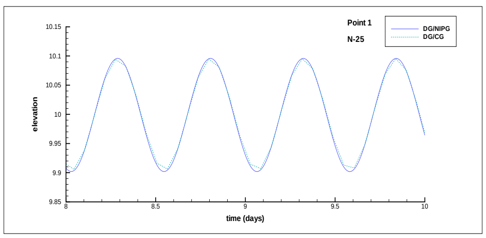
for $\alpha = .000140518917083$, where the tidal forcing function at $\zeta(x_L,t)$ is linearly ramped up over 2 days time. We run each simulation for $T = 12$ days with a time step of $dt = .25$ seconds on a mesh of $N = 25$ points, and a time step of $dt = .05$ on meshes of $N = 50, 100$ points.

In Figure 1(a), we compare the overlapping solutions for elevation ζ of each method at point $x = 800$ for the time length of the simulation on 25 nodes. A more detailed picture from time $t = 8$ to 10 days of the elevation is contained in Figure 1(b) and of the velocity in Figure 1(c).

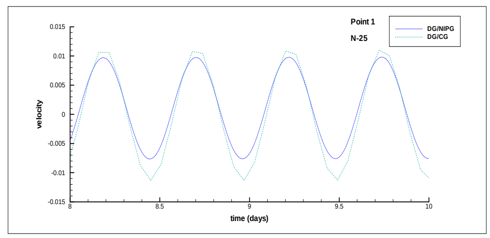
Similarly, in Figures 2 and 3, we compare the solutions for meshes of $N = 50$ and $N = 100$ respectively. Figures 4 through 6 display similar data at point $x = 3600$ in the mesh. As the number of elements increases, the methods converge to the same solutions.



(a) N=25 elevation at $x = 800$



(b) elevation for t=(8,10)

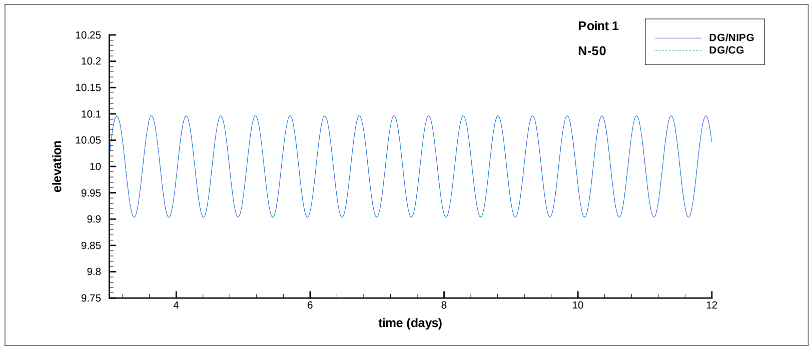


(c) velocity for t=(8,10)

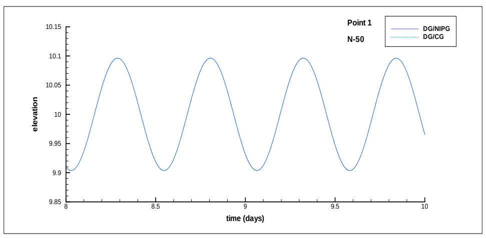
FIG. 1. *25 nodal points approximate solution to shallow water test case 2 at* $x = 800$ *for the discontinuous and continuous Galerkin method as well as the discontinuous Galerkin and NIPG formulation.*

6. Conclusions. In this paper, we have presented two formulations for the discretization of a simplified model of the two-dimensional shallow water equations. Both finite element methods utilize discontinuous approximating spaces for the primitive continuity equation. This DG approach has a number of useful properties, including adaptivity as well as the ability to incorporate non-conforming meshes and stablity post-processing. The first method utilizes continuous approximating spaces to discretize the continuous momentum equation and is useful when local conservation is important. The latter method discretizes the momentum equation via the NIPG method allowing for the flexibility of a DG method applied to both equations. We have presented an *a priori* error estimates and demonstrated "optimal" numerical convergence of order h^2 with linear finite elements for both methods.

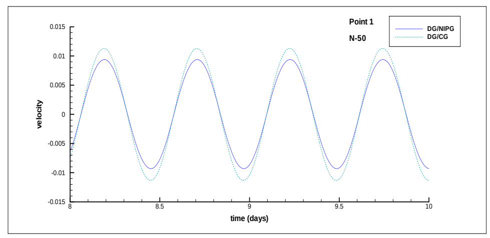
Good agreement between the two methods has been obtained for a particular case study modeling tidal fluctuations. Convergence to the same smooth solution was demonstrated as the mesh size was increased for this test problem. However, the combined DG and NIPG method results were particularly sensitive to the choice of parameters utilized in the simulation. We intend to investigate whether this is a function of the NIPG method specifically, or more generally the application of DG methods to the momentum equation.



(a) N=50 elevation at $x = 800$



(b) elevation for t=(8,10)



(c) velocity for t=(8,10)

FIG. 2. *50 nodal points approximate solution at* $x = 800$ *for the discontinuous and continuous Galerkin method as well as the discontinuous Galerkin and NIPG formulation.*

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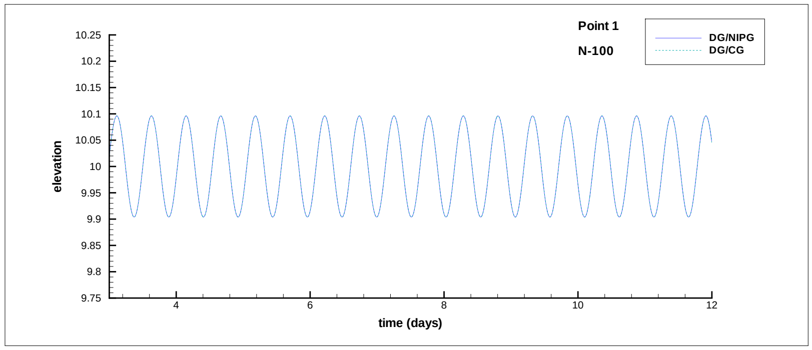
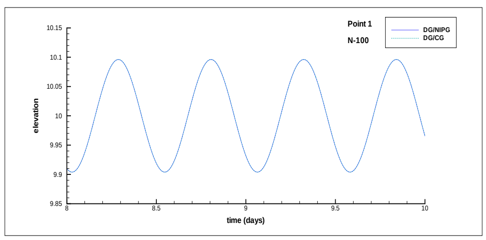
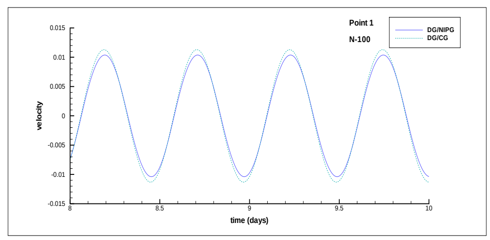


(a) N=100 elevation at $x = 800$



(b) elevation for t=(8,10)



(c) velocity for t=(8,10)

FIG. 3. *100 nodal points approximate solution at $x = 800$ for the discontinuous and continuous Galerkin method as well as the discontinuous Galerkin and NIPG formulation.*

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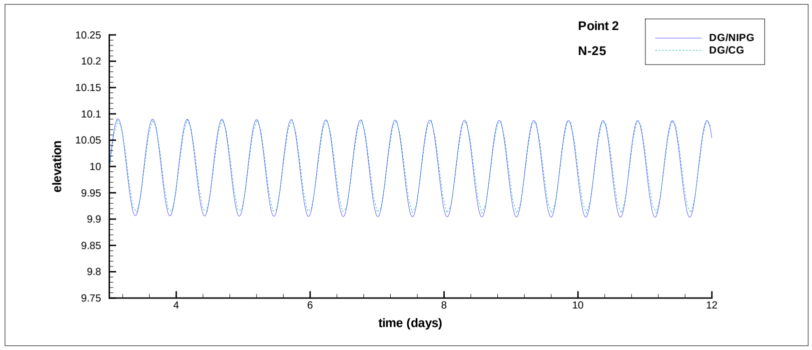
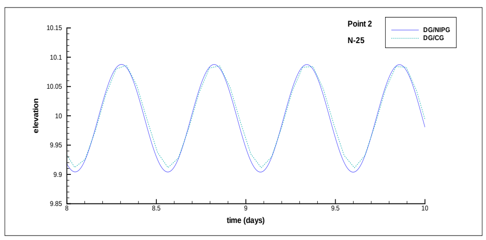
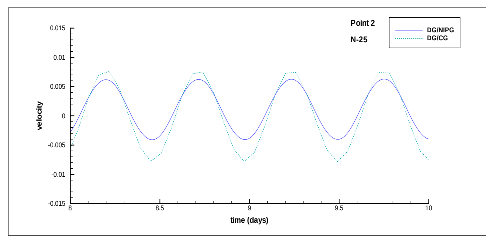


(a) N=25 elevation at $x = 3600$



(b) elevation for t=(8,10)



(c) velocity for t=(8,10)

FIG. 4. *25 nodal points approximate solution at $x = 3600$ for the discontinuous and continuous Galerkin method as well as the discontinuous Galerkin and NIPG formulation.*

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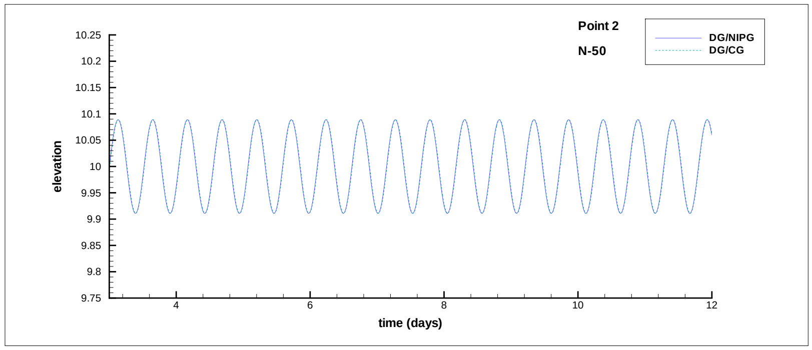
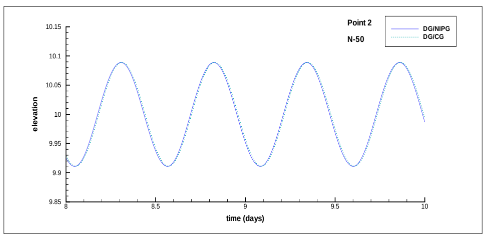
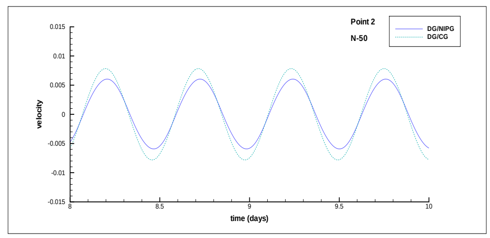


(a) N=50 elevation at $x = 3600$



(b) elevation for t=(8,10)



(c) velocity for t=(8,10)

FIG. 5. *50 nodal points approximate solution at* $x = 3600$ *for the discontinuous and continuous Galerkin method as well as the discontinuous Galerkin and NIPG formulation.*

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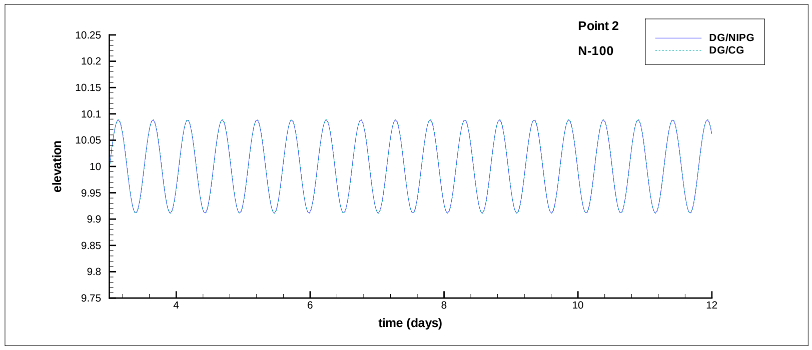
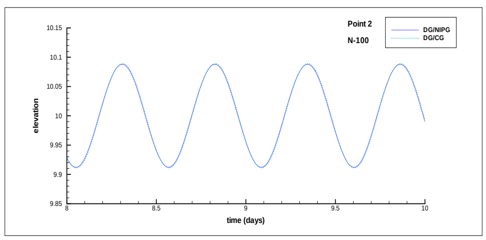
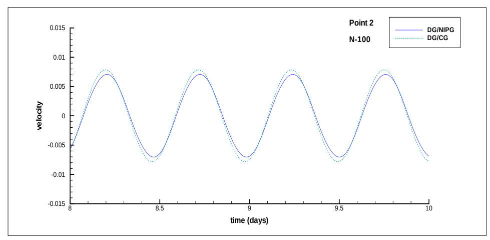


(a) N=100 elevation at $x = 3600$



(b) elevation for t=(8,10)



(c) velocity for t=(8,10)

FIG. 6. *100 nodal points approximate solution at $x = 3600$ for the discontinuous and continuous Galerkin method as well as the discontinuous Galerkin and NIPG formulation.*

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